Mike O'Sullivan
Department of Mathematics
San Diego State University

Spring 2017

Math 720: Commutative Algebra and Algebraic Geometry
Homework 3-Exam prep

In class we proved our main result so far, the Hilbert Basis Theorem (HBT) ((1)-(3) below), but we also developed a finer understanding of it due to Buchberger. By assuming we have a term order, we get a division algorithm in $k[x_1, \ldots, x_n]$, and we get a better way of describing ideals, determining ideal membership and computing as listed in the items below.

(1) A ring R satisfies the ascending chain condition if and only if every ideal in R is finitely generated.

(2) If a ring R satisfies the ACC then so does $R[y]$ [HBT].

(3) Consequently, every ideal in $k[x_1, \ldots, x_n]$ is finitely generated.

(4) Given a term order $<$, each polynomial has a well defined leading term. We have an algorithm for dividing a polynomial by another, or by a sequence of several polynomials.

(5) Given an ideal I, the ideal of leading terms $\langle LT(I)\rangle = \langle LT(f) : f \in I\rangle$ (a monomial ideal) is finitely generated, by HBT. Let $x^{\alpha_1}, \ldots, x^{\alpha_s}$ generate it. We may assume this set is minimal: no x^{α_i} divides x^{α_j} for $j \neq i$.

(6) Let $g \in I$ have $LT(g_i) = x^{\alpha_i}$. The ideal I is generated by the polynomials g_i. This is a minimal Groebner basis. Replacing each g_i by its remainder upon division by the others gives a reduced Groebner basis.

(7) Given an arbitrary generating set we may construct a GB using Buchberger's algorithm. The algorithm proceeds iteratively. Roughly speaking, it computes a syzygy polynomial of two polynomials in the generating set $S(f, h)$, then divides $S(f, h)$ by the generating set to get a remainder. If the remainder is nonzero it is added to the generating set. The algorithm proceeds with this enlarged generating set.

(8) When all syzygy polynomials have remainder 0 the generating set is a GB. The algorithm terminates in a finite number of steps.

In the coming week, in preparation for the first test, let's focus on computation. I added code for the division algorithm in the Sage folder. It's worthwhile doing some computation by hand, and some with Sage. Here is a list of problems that are computational, using the theory described above. (These are from the 2nd edition, your edition may be slightly different, just choose the computational problems.) Do enough to be confident. Class will be devoted to resolving any uncertainties about these problems and doing the two problems I wrote below.

- II.2 #1, 2, 11
- II.3 # 5, 6, 7
- II.5 #1-3, 7, 8, 17
- II.6 #5, 9
- II.7 #2, 3.

Please work the problems above before class on Tuesday, and bring any questions you have. Start the problems below as well.

Problem 1: Let $f = xy - z$, $g = y - z$, let $I = \langle f, g \rangle$.

(1) Check that we do have a generating set for the lex term order with $x > y > z$.

(2) For other term orders (switching the order of the indeterminates, or using grlex rather than lex) find a GB.

(3) Continuing with lex $x > y > z$, identify a basis for $k[x,y,z]/I$ and write down a general element of $k[x,y,z]/I$.

(4) Explain how to compute in $k[x,y,z]/I$.

(5) Show that the associate variety is the union of two lines L_1 and L_2. Show that $\mathbb{I}(L_1)$ contains I and similarly for L_2.

(6) Analyze the ring map

$$k[x,y,z]/I \longrightarrow k[x,y,z]/\mathbb{I}(L_1) \times k[x,y,z]/\mathbb{I}(L_1)$$

Is it injective? Is it surjective?

Problem 2: Repeat the problem above with $g = x^2 - x$.