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Math 627A: Modern Algebra I

Homework I

Problem 1: (see Rotman ex. 26)

- **(a)** If $\phi : R \to S$ is an isomorphism of rings, then $\phi^{-1} : S \to R$ is also.
- **(b)** If $\phi : R \to S$ and $\psi : S \to T$ are ring homomorphisms, then so is their composite, $\psi \circ \phi : R \to T$.

Problem 2: (see Rotman ex. 32)
In a ring R, we say s is an *associate* of r when there is a unit $u \in R$ such that $s = ur$.

- **(a)** Show that the relation of being associates is an equivalence relation.
- **(b)** What is the equivalence class of 1_R under the associate relation?
- **(c)** For $r \in R$, and u a unit in R, $\langle r \rangle = \langle ur \rangle$. Conclude that
  - If an ideal I contains a unit u then $I = R$.
  - If r and s are associates, they generate the same principal ideal.
- **(d)** If R is a domain and $r, s \in R$, then $\langle r \rangle = \langle s \rangle$ if and only if $s = ur$ for some unit $u \in R$.
- **(e)** For the ring $\mathbb{Z}/30$, find two elements, such that $\langle r \rangle = \langle s \rangle$, but s and r are not associates.

Problem 3:
Let R be an integral domain. Let M be a subset of $R \setminus \{0\}$ that contains 1 and is closed under multiplication.

We discussed the following in class:

1. The relation on $R \times M$ defined by
$$(a_1, m_1) \sim (a_2, m_2) \quad \text{when} \quad a_1 m_2 = a_2 m_1$$
is an equivalence relation.

2. Let $[a, m]$ denote the equivalence class of (a, m). The operations
   - $[a_1, m_1] + [a_2, m_2] := [a_1 m_2 + a_2 m_1, m_1 m_2]$, and
   - $[a_1, m_1] \star [a_2, m_2] := [a_1 a_2, m_1 m_2]$,

   are well defined.

3. The set $R \times M/\sim$ with these operations is a ring with additive identity $[0, 1]$ and multiplicative identity $[1, 1]$. We denote this ring $M^{-1}R$.

The ring $M^{-1}R$ is often called a localization of R. In this exercise we characterize all localizations of $\mathbb{Z}$.

(a) Let $A = \{a_1, a_2, \ldots, a_t\} \subset \mathbb{Z} \setminus \{0\}$. What is the smallest multiplicatively closed subset of $\mathbb{Z}$ containing A and 1? What if A is an infinite subset of $\mathbb{Z} \setminus \{0\}$?
We will abuse notation and write $A^{-1}\mathbb{Z}$ for the localization due to the smallest multiplicatively closed subset of $\mathbb{Z} \setminus \{0\}$ containing A.

(b) Suppose $M \subset N \subset \mathbb{Z} \setminus \{0\}$. Show that there is an injective ring homomorphism $M^{-1}\mathbb{Z} \to N^{-1}\mathbb{Z}$.

(c) Any localization of $\mathbb{Z}$ is of the form $P^{-1}\mathbb{Z}$ where P is a subset of the set of primes in $\mathbb{N}$. [You need to identify the set P for a given M and show that $P^{-1}\mathbb{Z} \cong M^{-1}\mathbb{Z}$].

Due: Friday, September 23rd. You are encouraged to work together to analyze and solve the problem, but please write your solutions individually.