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Math 627A: Modern Algebra I

Homework 4

Problem 1: (Rotman 63, 64, 67)

(a) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ be a polynomial with integer coefficients. If r/s is a rational root of $f(x)$ with r/s in lowest terms, then $r \mid a_0$ and $s \mid a_n$.

(b) What can you conclude in each of the following special cases?

- $a_n = 1$.
- $a_0 = 1$.
- $a_{n-1} = 0$.

(c) Let $f(x) = x^4 + Ax^2 + B$. Give necessary and sufficient conditions for $f(x)$ to have a nontrivial factor. Consider the case when $f(x)$ has a linear factor and the case when it has quadratic factors separately.

Problem 2: (Rotman 64) Test whether the following polynomials factor over the rationals.

(a) $6x^3 - 3x - 18$

(b) $3x^2 - 7x + 5$.

(c) $x^3 - 9x - 9$.

(d) $16x^2 + 5x + 32$

(e) $2x^6 + 5x + 35$.

Problem 3: (Rotman 70, 71a)

(a) Let $g(x) = x^3 + qx + r$ and define $R = r^2 + 4q^3/27$. Recall that we wrote a root u of $g(x)$ as $u = y + z$ and found that $y^3 = (-r + \sqrt{R})/2$ and $z = -3q/y$. Prove that

$$z^3 = \frac{1}{2}\left(-r - \sqrt{R}\right)$$

(b) Find the roots of the polynomial $f(x) = x^3 - 3x + 1$.

Problem 4: (Rotman 76) Prove that a field of characteristic p is perfect if and only if each element has a pth root.

Problem 5: (Rotman 72)

(a) Let E/F be an extension of fields and let $\alpha, \beta \in E$ be algebraic over F. Prove that $F(\alpha, \beta)$ is an algebraic extension over F. (This shows that $\alpha + \beta$, $\alpha\beta$ and α^{-1} are all algebraic over F.)

(b) Let E/F be an extension of fields and let

$$K = \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$$

Show that K is a subfield of E containing F.

(c) Let $\bar{\mathbb{Q}}$ be the set of elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. These are called the algebraic numbers. Show that $\bar{\mathbb{Q}}$ is infinite dimensional over F. (Hint: $\sqrt[n]{2} \in \bar{\mathbb{Q}}$.)

Due: Wednesday, October 26, class time. You are encouraged to work together to analyze and solve the problem, but please write your solutions individually.