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Math 627A: Modern Algebra I

Homework 5

Problem 1: Let A_1, A_2, B_1, B_2 be normal subgroups of a group G with $B_1 \leq A_1$ and $B_2 \leq A_2$ and finally $A_1 \cap A_2 = \{1\}$. Then $B_1 B_2$ is normal subgroup in $A_1 A_2$ and there holds

$$(A_1 A_2)/(B_1 B_2) \cong (A_1/B_1) \oplus (A_2/B_2).$$

Problem 2: If G is a free Abelian group of ranks r and s, then $r = s$.

Prove this in the following way: Let $\varphi : \mathbb{Z}^r \longrightarrow \mathbb{Z}^s$ be an isomorphism, and let α and β be the natural embeddings of $\mathbb{Z}^r$ (resp. $\mathbb{Z}^s$) into the respective direct sums of copies of $\mathbb{Q}$.

- **(a)** Show that for every $x \in \mathbb{Q}^n$ there exists $z \in \mathbb{Z}$ such that $zx \in \mathbb{Z}^n$.
- **(b)** Define $\overline{\varphi} : \mathbb{Q}^r \longrightarrow \mathbb{Q}^s$ by $x \mapsto \frac{1}{z}\varphi(zx)$ where z is the number that you found in **(a)**. Show that this mapping is well-defined.
- **(c)** Show that $\overline{\varphi}$ is additive and (hence) $\mathbb{Z}$-linear; then show that $\overline{\varphi}$ is $\mathbb{Q}$-linear.
- **(d)** Show that $\overline{\varphi}$ is one-to-one. Show that $\overline{\varphi}$ is onto.
- **(e)** Conclude that $r = s$.

Problem 3: Let G be a group. For $a, b \in G$ define the commutator $[a,b] := aba^{-1}b^{-1}$ of a and b. For arbitrary subgroups U, V of G define $[U,V] := \langle [u,v] \mid u \in U, v \in V \rangle$. Now show the following:

(a) If U, V are normal subgroups of G, then so is $[U,V]$.

(b) $[G,G]$ is the smallest normal subgroup of G for which the quotient group is abelian.

(c) Setting $G^{(0)} := G$ and $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$ for all $i \in \mathbb{N}$, we find that G is solvable if and only if there exists $n \in \mathbb{N}$ such that $G^{(n)} = \{1\}$.

(d) For $n \geq 5$ let U be a subgroup of S_n and N a normal subgroup of U for which U/N is abelian. Show that if U contains all 3-cycles of S_n, then also N will contain these.
Hint: If $a, b, c, d, e \in \{1, \ldots, n\}$ are distinct elements, then there holds the equation

$$(a,b,c) = (a,b,d)(c,e,a)(d,b,a)(a,e,c).$$

(e) Show that this implies that the symmetric group S_n is not solvable for $n \geq 5$.

Due: Monday, November 14, class time. You are encouraged to work together to analyze and solve the problem, but please write your solutions individually.