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Math 627A: Modern Algebra I

Homework 6

Problem 1: Let F be a field of characteristic $p \neq 0$ and let $a \in F$.

(a) Show that the polynomial $f = x^p - x - a \in F[x]$ is separable.

(b) Show that either f splits or it is irreducible. Break this into two steps:

- If f possesses a root in F then f already splits over F.
- If f is reducible over F then it has a root in F.

(c) Suppose that f is irreducible. What is the Galois group of the splitting field of f over F?

Hint: In **(b)**, suppose α is a root of f in F, and consider the element $\alpha + \gamma$ with $\gamma \in \mathbb{F}_p$ as a further candidate for a root. Then assume a factorization $f = gh$ and look at one of the coefficients of g.

Problem 2: Show that each of the following extensions is a Galois extension of $\mathbb{Q}$. Find the according Galois group.

(a) $\mathbb{Q}(i + \sqrt{2})$

(b) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

Problem 3: Show that $\mathbb{Q}(i\sqrt{5})/\mathbb{Q}$ and $\mathbb{Q}((1+i)\sqrt[4]{5})/\mathbb{Q}(i\sqrt{5})$ are normal, but $\mathbb{Q}((1+i)\sqrt[4]{5})/\mathbb{Q}$ is not normal.
(See also Rothman Ex. 88)

Problem 4: Prove: If H is an intermediate field of the finite Galois extension G/F then H/F is normal if and only if $\sigma(H) \subseteq H$ for all $\sigma \in \mathrm{Aut}(G/F)$.

Due: Monday, December 5, class time. You are encouraged to work together to analyze and solve the problem, but please write your solutions individually.