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Math 627A: Modern Algebra I

Homework II

Please read the following problems and their solutions in Ash's text. Many of them are routine, and others have been covered in class to some extent.

- §1.1 pr. 2-11.
- §1.2 pr. 1-8.
- §1.3 pr. 1-12.
- §1.4 pr. 1-9.
- §1.5 pr. 1-8.

Let G be a group. The following groups and subgroups are important. You should be able to establish these results.

- $Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$ is a normal subgroup of G.
- The centralizer of $a \in G$, $C(a) = \{g \in g : ga = ag\}$ is a subgroup of G containing a. $Z(G) = \bigcap_{a \in A} C(a)$.
- Let H be a subgroup of G. The normalizer of H, $N_H = \{x \in G : x^{-1}Hx = H\}$ is a subgroup of H containing H. H is normal in N_H. Any subgroup K of G that contains H as a normal subgroup is contained in N_H. (If $N \trianglelefteq K \leq G$ then $K \leq N_H$.)
- The set of automorphisms of G, $\mathrm{Aut}(G)$, is a group.
- The set of inner automorphisms of G, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
- If G is *abelian*, $\mathrm{Tor}(G) = \{a \in G : \mathrm{ord}(a) \text{ is finite }\}$ is a normal subgroup of G and $G/\mathrm{Tor}(G)$ has no elements of finite order.
- (harder) The commutator subgroup of G, is the subgroup G' of G generated by $S = \{aba^{-1}b^{-1} : a, b \in G\}$. G' is normal in G and G/G' is abelian.
  If N is normal in G and $N \cap G' = \{e\}$ then $N \subset Z(G)$ and $Z(G/N) \cong Z(G)/N$.

Problem 1: Let H be a subgroup of G. Show that:
(a) N_H is a subgroup of G;
(b) H is normal in N_H;
(c) if K is a subgroup of G and H is normal in K then K is a subgroup of N_H.

Problem 2: For $a \in G$ let f_a be the inner automorphism defined by a and consider the function $F : a \mapsto f_a$.

$$\begin{aligned} f_a : G &\longrightarrow G \\ g &\longmapsto aga^{-1} \end{aligned} \qquad\qquad \begin{aligned} F : G &\longrightarrow \mathrm{Aut}(G) \\ a &\longmapsto f_a \end{aligned}$$

Clearly $\mathrm{im}(F) = \mathrm{Inn}(G)$.
(a) Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
(b) Show that F is a homomorphism and that $\mathrm{im}(F) \cong G/Z(G)$.

Problem 3: Let G be a group generated by elements a and b such that $a^4 = e$, $a^2 = b^2$, and $ba = a^3 b$. Prove that G is isomorphic to the Quaternion group.

Problem 4: Some normal subgroups.
(a) Let H be the intersection of all subgroups of G of order n.
Prove that H is normal in G.
(b) Let H be a subgroup of G and $N = \bigcap_{a \in G} a^{-1} H a$. Prove that N is normal in G.

Problem 5: G is called metabelian if it has a normal subgroup N such that N and G/N are abelian. Show the following: (a) S_3 is metabelian; (b) every subgroup of a metabelian is metabelian; (c) every homomorphic image of a metabelian group is metabelian.
[Hint: The 2nd isomorphism theorem.]

Problem 6: Suppose G is abelian and $f : G \to \mathbb{Z}$ is surjective. Let K be the kernel. Show G has a subgroup H isomorphic to $\mathbb{Z}$ and $G \cong H \oplus K$.

Problem 7: Consider the following subgroups of $\mathrm{Gl}(F, n)$:

$$\mathrm{Sl}(F, n) \qquad \mathrm{Diag}(F, n) \qquad F^* I_n$$

(a) Show that $\mathrm{Sl}(F, n)$ and $F^* I_n$ are normal in $\mathrm{Gl}(F, n)$ but that $\mathrm{Diag}(F, n)$ is not, except in one very special case.
(b) Is it true that $\mathrm{Gl}(F, n)$ is the product of $\mathrm{Sl}(F, n)$ and $F^* I_n$? The answer is subtle—it depends on the field and on n!
To get started you might want to try some small fields, like $\mathbb{F}_3$, using a computer algebra system.