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Math 627B: Modern Algebra II

Homework IV

Reading: Ash, chapter 2. Ignore anything with non-commutative rings. Work the problems below. Ask me if you have any trouble with them!

Section	Problems
2.1	#1, 2, 4, 5
2.2	# 1, 6, 7, 8
2.3 (not 2.3.3)	#5
2.4	# 1, 2, 7, 8, 9
2.6	# (1-6), 7, 8
2.7	# 9
2.8	# 1-8
2.9	# 4, 7, 8

Problem 1: Ideals and localization:
Let R be an integral domain and let S be a multiplicatively closed subset of R.
We will consider R as a subset of $S^{-1}R$ via the embedding $r \to r/1$.
(a) Show that $S^{-1}I = \{a/s : a \in I, s \in S\}$ is an ideal in $S^{-1}R$.
(b) Show that $S^{-1}I = S^{-1}R$ iff $I \cap S \neq \emptyset$.
(c) Let J be an ideal of $S^{-1}R$. Show that $J \cap R$ is an ideal in R.
From 1a - c we have a function ϕ from the set of ideals in $S^{-1}R$ to the set of ideals in R and a function θ from the set of ideals in R to the set of ideals in $S^{-1}R$.
(d) Show that θ is surjective: That is show that every ideal in $S^{-1}R$ is $S^{-1}I$ for some ideal I in R. (Hints: If J is an ideal in $S^{-1}R$ then $S^{-1}(J \cap R) = J$. An element of J may be written a/s for $a \in R$ and $s \in S$).
(e) Show that ϕ and θ respect interesections $\phi(J \cap J') = \phi(J) \cap \phi(J')$ and similarly $\theta(I \cap I') = \theta(I) \cap \theta(I')$. (Don't bother with sums.)
(f) Show that these functions give a 1-1 correspondence between prime ideals of $S^{-1}R$ and prime ideals of R not meeting S.

Problem 2: Localization in $\mathbb{Z}$ and $\mathbb{C}[x]$.
(a) Let $S = \{30^i : i \in \mathbb{N}_0\}$. Verify that S is multiplicatively closed in $\mathbb{Z}$. Identify all of the prime ideals in $S^{-1}\mathbb{Z}$.
(b) Let $S = \{(x^3 - x)^i : i \in \mathbb{N}_0\}$. Verify that S is multiplicatively closed in $\mathbb{C}[x]$. Identify all of the prime ideals in $S^{-1}\mathbb{C}[x]$.
(c) Under what conditions on S does $S^{-1}\mathbb{Z}$ have just one maximal ideal?

Problem 3: Let $f = y^2 - x^3$ and let $g = y^2 - x^3 - x^2$.
(a) Show that f and g are irreducible in $\mathbb{C}[x, y]$.
(b) Show that the $R = \mathbb{C}[x, y]/\langle f \rangle$ is not a UFD. Do the same for g.
(c) Show that localizing doesn't help, $R_{\langle x,y \rangle}$ is not a UFD in either case (R defined by f or g).