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Math 627B: Modern Algebra II

Homework I

Problem 1: The radical of a polynomial. IVA §1.5 #14, 15. For how large a degree is Sage able to compute $f_{\rm red}$? Compare with the time it takes to factor in order to compute $f_{\rm red}$.

Problem 2: The variety of several polynomials in $\mathbb{C}[x]$. IVA §1.5 #16.

Problem 3: Sketching some varieties in $\mathbb{R}^n$: IVA §1.2 #4e, 5.

Problem 4: Varieties and non-varieties.
(a) Prove that a finite set of points is a variety. §1.2 #6.
(b) Prove that the punctured line is not a variety. §1/2 #8.
(c) Prove that the Cartesian product of varieties is a variety. §1.2 #15d.

Problem 5: The Euclidean Algorithm
Input $f, h \in F[x]$.
Output $r, u, v, y, z \in F[x]$ such that

1. (1) $s = fy + hz$ and s is some constant multiple of $\gcd(f, h)$;
2. (2) $fu = -hv$ is a constant multiple of $\mathrm{lcm}(f, h)$.

Initialize

$$M = \begin{bmatrix} r & u & v \\ s & y & z \end{bmatrix} = \begin{bmatrix} f & 1 & 0 \\ h & 0 & 1 \end{bmatrix}$$

Algorithm While $r \neq 0$ do

$$q \longleftarrow s//r \text{ (the polynomial quotient).}$$
$$M \longleftarrow \begin{bmatrix} -q & 1 \\ 1 & 0 \end{bmatrix} M$$

(a) Prove that at every iteration

$$\begin{bmatrix} u & v \\ y & z \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

and that $uz - yv = \pm 1$.
(b) Show that at termination y, z give the $\gcd(f, h)$ and u, v give the $\mathrm{lcm}(f, h)$ as claimed in the output statement. (You will need that u and v are coprime, see part (a).)