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Math 627B: Modern Algebra II

Homework IV

Due: Monday Oct 24, 2011

Reading: Ash, chapter 2. Ignore anything with non-commutative rings. Most of the material in 2.1, 2.2, and 2.5, plus some of 2.3 and 2.4 is in Hungerford's book. We mainly need 2.3, 2.4, 2.8, and—later in the course—2.6, 2.9. Ask me if you have any trouble with them!

- Basic properties of rings.
  - 2.1.1-2
  - 2.1.3 examples 5,6
  - 2.1.4-5 (can be proven by induction also)
  - 2.1 Probs 1,2,4,5
- Ideals, homomorphisms, quotient rings
  - 2.2.1-8
  - 2.2 Problems 6,7,8
- Isomorphism Theorems
  - 2.3.1-5
  - 2.3.6-7 Direct products and the Chinese Remainder theorem.
  - 2.3 Problems 2,3,4,5,6
  - See also 2.5 problems 7,8
- 2.4 Maximal and prime ideals
  - Problems 1,2,7,8,9
- 2.5 Polynomial rings (nothing new here)

- 2.8 Rings of Fractions
  - 2.8.1-2 Definition of the ring of fractions (aka localization) $S^{-1}R$.
  - R is a subring of $S^{-1}R$ when R is an integral domain.
  - Problems 1-4, 7, 8
- 2.6 Unique Factorization
- 2.9 Irreducible Polynomials

Section	Problems
2.1	#1, 2, 4, 5
2.2	# 1, 6, 7, 8
2.3 (not 2.3.3)	#2, 3, 4, 5, 6
2.5	#7, 8
2.4	# 1, 2, 7, 8, 9
2.8	# 1-8
2.6	# (1-6), 7, 8
2.7	# 9
2.9	# 4, 7, 8

Problem 1: An ideal I in R is *radical* when $a^n \in I$ implies that $a \in I$. An element $a \in R$ is *nilpotent* when $a^n = 0$ for some positive integer n.

(a) Find all nilpotent elements of $\mathbb{Z}/36$.

(b) Identify all radical ideals in $\mathbb{Z}$. To be more explicit, under what conditions is $\langle n \rangle$ a radical ideal? Justify your answer.

(c) Prove that I is radical iff R/I has no nonzero nilpotent elements.

Problem 2: Homomorphisms and ideals. Let $\phi : R \longrightarrow S$ be a homomorphism.

(a) If J is an ideal in S, show that $\phi^{-1}(J)$ is an ideal in R.

(b) If J is prime in S, show that $\phi^{-1}(J)$ is prime in R.

(c) Suppose now that ϕ is surjective. For I and ideal in R, we showed that $\phi(I)$ is an ideal of S. Show that $\phi^{-1}(\phi(I)) = I + K$ where $K = \ker \phi$. In particular, if I contains K, then $\phi^{-1}\phi(I) = I$.

Problem 3: Computation in quotient rings using Grobner bases.
Let $I = \langle x - y^2, x^2y - x\rangle$ in $k[x,y]$ where k is a field. Consider the four term orders lex and glex with $x > y$ and $y > x$. Recall the Groebner bases for I with respect to these four orders from the previous assignment. For each term order compute $(x+y)^2$ using the standard basis for $k[x,y]/I$ in that term order.

Problem 4: Ideals and localization:
Let R be an integral domain and let S be a multiplicative subset of R.
We will consider R as a subset of $S^{-1}R$ via the embedding $\phi : R \longrightarrow S^{-1}R$ which takes r to $r/1$.

- **(a)** Let $S^{-1}I = \{a/s : a \in I, s \in S\}$. Show $S^{-1}I$ is an ideal in $S^{-1}R$.
- **(b)** Show that $S^{-1}I = S^{-1}R$ iff $I \cap S \neq \emptyset$.
- **(c)** Let J be an ideal of $S^{-1}R$. Show that $J \cap R$ is an ideal in R.

Parts a - c show we have a function from the set of ideals in $S^{-1}R$ to the set of ideals in R given by $J \mapsto J \cap R$ and a function from the set of ideals in R to the set of ideals in $S^{-1}R$ given by $I \mapsto S^{-1}I$.

- **(d)** Show that $I \mapsto S^{-1}I$ is surjective: That is, show that every ideal in $S^{-1}R$ is $S^{-1}I$ for some ideal I in R. (Hints: If J is an ideal in $S^{-1}R$ then an element of J may be written a/s for $a \in R$ and $s \in S$. Show that $S^{-1}(J \cap R) = J$.)
- **(e)** Show that these two maps of ideals respect intersections. For example, $S^{-1}(I \cap I') = S^{-1}(I) \cap S^{-1}(I')$.
- **(f)** The map $I \mapsto S^{-1}I$ is not injective. Show that it is injective on prime ideals. Conclude that the functions $J \mapsto J \cap R$ and $I \mapsto S^{-1}I$ give a 1-1 correspondence between prime ideals of $S^{-1}R$ and prime ideals of R not meeting S.

Problem 5: Localization in $\mathbb{Z}$ and $\mathbb{C}[x]$.

- **(a)** Let $S = \{30^i : i \in \mathbb{N}_0\}$. Verify that S is multiplicatively closed in $\mathbb{Z}$. Identify all of the prime ideals in $S^{-1}\mathbb{Z}$.
- **(b)** Let $S = \{(x^3 - x)^i : i \in \mathbb{N}_0\}$. Verify that S is multiplicatively closed in $\mathbb{C}[x]$. Identify all of the prime ideals in $S^{-1}\mathbb{C}[x]$.
- **(c)** Under what conditions on S does $S^{-1}\mathbb{Z}$ have just one maximal ideal?