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Math 627B: Modern Algebra II
Homework VIII

Due Tu 5/1, 2012.

Problem 1: Infinite algebraic extensions.

(a) Find a sequence of fields $\mathbb{Q} \le F_1 \le F_2 \le \cdots$ such that each F_i is a normal extension of $\mathbb{Q}$. Justify your answer. [See 6.3#5 for inspiration.]

(b) Find a sequence of fields $\mathbb{Q} \le F_1 \le F_2 \le \cdots$ such that each F_{i+1} is a normal extension of F_i but not of F_{i-1}. [See 6.3#9 for inspiration.]

Problem 2: Let E be a Galois extension of F. Let E_1 and E_2 be two intermediate fields and let $H_i = \mathrm{Gal}(E/E_i)$ for $i = 1, 2$. Let $E_1 \vee E_2$ be the compositum of E_1 and E_2—the smallest field containing them both. Let $H_1 \vee H_2$ be the smallest group containing H_1 and H_2.

(a) Explain why $\mathrm{Gal}(E/E_1 \vee E_2) = H_1 \cap H_2$. [One direction is easy, the other just requires an understanding of the elements of $E_1 \vee E_2$.]

(b) Explain why $\mathrm{Fix}(H_1 \vee H_2) = E_2 \cap E_2$. [Again, one direction is easy, the other just requires an understanding of the elements of $H_1 \vee H_2$.]

(c) Now use Galois' main theorem to show that $E_1 \vee E_2 = \mathrm{Fix}(H_1 \cap H_2)$ and $H_1 \vee H_2 = \mathrm{Gal}(E_1 \cap E_2)$.

Problem 3: Let $E = \mathbb{Q}(i + \sqrt{2})$.

(a) Show that i and $\sqrt{2}$ are both in E.

(b) Find the minimum polynomial of $i + \sqrt{2}$.

(c) Show this polynomial splits in E.

(d) Find the Galois group and map out the relationship between subfields and subgroups for $E/\mathbb{Q}$.

Problem 4: Let E be the splitting field of $x^6 - 2$

(a) Explain why the Galois group is isomorphic to D_{12}, the dihedral group on 6 elements.

(b) Find the Galois group and map out the relationship between subfields and subgroups for $E/\mathbb{Q}$.