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Math 623: Matrix Analysis
Homework I

Problem 1: Commuting matrices.

(a) Show by example that commuting is not a transitive relation. It is possible for A to commute with B and B with C, while A does not commute with C.

(b) Let S be a fixed invertible matrix. Show that $\{SDS^{-1} : D \text{ is diagonal }\}$ is a commuting family of matrices. Any two matrices in the set commute with each other.

Problem 2: Elementary matrices. Let $E(i,j,a)$ be the matrix which is 1 on the diagonal, a in position (i,j) and 0 elsewhere. Let $T(\alpha,\beta)$ the transposition matrix, which is 0 in all positions except for following positions, where it is 1: (α,β), and (β,α) and (i,i) for $i \notin \{\alpha,\beta\}$.

(a) Show that $E(i,j,a)$ and $E(i,k,b)$ commute.

(b) Show that

$$T(\alpha,\beta)E(2,1,a)T(\alpha,\beta) = \begin{cases} E(1,2,a) & \alpha=1, \beta=2 \\ E(2,\beta,a) & \alpha=1, \beta>2 \\ E(\beta,1,a) & \alpha=2, \beta>2 \\ E(2,1,a) & \alpha,\beta>2 \end{cases}$$

Problem 3: Define a $n \times n$ matrix A to be r-lower triangular, for $n \leq r \leq n$, when $a_{i,j} = 0$ for all $j > i + r$. One defines r-upper triangular, similarly. So, 0-lower triangular is equivalent to lower triangular and (-1)-lower triangular is what we might call strictly lower triangular (0 on the diagonal).

(a) Prove that the product of an r-lower triangular and an s-lower triangular matrix is $(r+s)$-lower triangular.

(b) If A is an $n \times n$ strictly lower triangular matrix then $A^n = 0$.

Problem 4: We say that a matrix A is *nilpotent* when $A^k = 0$ for some $k \geq 0$. The minimal such k is the *index of nilpotency.*

(a) If A is nilpotent, prove that SAS^{-1} is also nilpotent and it has the same degree of nilpotency as A does.

(b) Can you say anything about the index of nilpotency for an r-upper triangular matrix?