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Math 623: Matrix Analysis
Homework 4

Problem 1: Orthogonality

(a) In $\mathbb{R}^n$, what angle does the vector $(1, 1, \dots, 1)$ make with the coordinate axes? (You will have to describe it using an inverse trig function.)

(b) In $\mathbb{R}^n$, show that $x - y$ is orthogonal to $x + y$ iff $\|x\| = \|y\|$.

(c) In $\mathbb{C}^n$ show that $\|x\| = \|y\|$ does not necessarily imply $x - y$ is orthogonal to $x + y$. Use $n = 1$ and $x = 1 + i$ and $y = 1 - i$.

(d) In $\mathbb{C}^n$ show that $x - y$ is orthogonal to $x + y$ iff $\|x\| = \|y\|$ and $\langle x, y \rangle$ is real.

Problem 2: Apply Gram-Schmidt to the following matrix A to factor it as $A = QR$ with Q having orthonormal columns. Find Q explicitly. You may compute R explicitly or write it as a product of matrices. Be careful, since we are working over $\mathbb{C}$.

$$\begin{bmatrix} 2 & -2 & 0 \\ 2i & -3i & -2i \\ 2i & -2i & 2i \\ 2 & -1 & 4 \end{bmatrix}$$

Problem 3: On the vector space of polynomials, $\mathbb{R}[x]$, consider the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$$

Recall that we showed that

$$\langle x^i, x^j \rangle = \begin{cases} 0 & \text{when } i + j \text{ is odd} \\ 2/(i + j + 1) & \text{when } i + j \text{ is even} \end{cases} \tag{1}$$

The Legendre polynomials $P_n(x)$ (for $n \in \mathbb{N}_0$) are orthogonal polynomials relative to this inner product. Each P_n has degree n, and is not normalized (length 1), but satisfies $\langle P_n, P_n \rangle = 2/(2n + 1)$. The first 5 Legendre polynomials P_0, P_1, P_2, P_3, P_4 are

$$1, x, (3x^2 - 1)/2, (5x^3 - 3x)/2, (35x^4 - 30x^2 + 3)/8$$

(a) Verify for yourself that $1, x, 3x^2 - 1$ are orthogonal.

(b) Use Gram-Schmidt to modify x^3 by multiples of P_0, P_1, P_2 to obtain a polynomial orthogonal to P_0, P_1, P_2. (Use (1) which shows x^3 is already orthogonal to x^i for i even, and use the value for $\langle P_n, P_n \rangle$ given above.) You should get a constant multiple of P_3.

(c) As in (b), use Gram-Schmidt to modify x^4 to obtain a polynomial orthogonal to P_0, P_1, P_2, P_3.