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Math 720: Commutative Algebra and Algebraic Geometry
Homework 1

Here is a list of problems to do. Some we will discuss in class. Those marked in bold **Turn in** should be turned in on Friday 2/3.

Problem 1: One dimension: polynomials in $\mathbb{C}[x]$ and their varieties.

- **(a) TURN in:** The radical of a polynomial. Do IVA §1.5 #14, 15a to show that $f_{\text{rad}} = \frac{f}{\gcd(f,f')}$.
- **(b) TURN in:** For the extended Euclidean algorithm (below). Prove that at every iteration
$$M_i \begin{bmatrix} a \\ b \end{bmatrix} = R_i$$
and that $u_i z_i - y_i v_i = \pm 1$. Show that at termination y, z give the $\gcd(a,b)$ and u, v give the $\operatorname{lcm}(a,b)$ as claimed in the "Produce" statement. (You will need that u and v are coprime, see part (a).)
- **(c)** The variety of several polynomials in $\mathbb{C}[x]$. IVA §1.5 #16

Problem 2: Varieties and non-varieties in $\mathbb{R}^n$:

- **(a)** Do a few examples in IVA §1.2, e.g. #4e, 5.
- **(b)** Prove that a finite set of points is a variety. §1.2 #6.
- **(c)** Prove that the punctured line is not a variety. §1/2 #8.
- **(d) TURN in:** Prove that the Cartesian product of varieties is a variety. §1.2 #15d.

Problem 3: An ideal I in R is *radical* when $a^n \in I$ implies that $a \in I$. An element $a \in R$ is *nilpotent* when $a^n = 0$ for some positive integer n.

- **(a)** Find all nilpotent elements of $\mathbb{Z}/600$.
- **(b) TURN in:** Identify all radical ideals in $\mathbb{Z}$. To be more explicit, under what conditions is $\langle n \rangle$ a radical ideal? Justify your answer.

(c) TURN in: Let $N = \{a \in R : a^n = 0 \text{ for some } n\}$ be the set of nilpotent elements in R. Prove that N is an ideal.

(d) Prove that I is radical iff R/I has no nonzero nilpotent elements.

Problem 4: Homomorphisms and ideals. Let $\varphi : R \longrightarrow S$ be a homomorphism.

(a) If J is an ideal in S, show that $\varphi^{-1}(J)$ is an ideal in R.

(b) If J is prime in S, show that $\varphi^{-1}(J)$ is prime in R.

(c) TURN in: Show that $\varphi^{-1}(\varphi(I)) = I + K$ where $K = \ker \varphi$. In particular, if I contains K, then $\varphi^{-1}\varphi(I) = I$.

The Extended Euclidean Algorithm
Input $a, b \in k[x]$.
Produce

$$M = \begin{bmatrix} u & v \\ y & z \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} r \\ s \end{bmatrix}$$

where $u, v, y, z, r, s \in k[x]$ such that

(1) $s = ay + bz$ and s is some constant multiple of $\gcd(a, b)$;

(2) $au = -bv$ is a constant multiple of $\operatorname{lcm}(a, b)$.

Initialize $i = 0$, and

$$M_0 = \begin{bmatrix} u_0 & v_0 \\ y_0 & z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$R_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Algorithm While $r_i \neq 0$ do

$$q_i \longleftarrow s_i // r_i \text{ (the polynomial quotient).}$$
$$M_{i+1} \longleftarrow \begin{bmatrix} -q_i & 1 \\ 1 & 0 \end{bmatrix} M_i$$
$$R_{i+1} \longleftarrow \begin{bmatrix} -q_i & 1 \\ 1 & 0 \end{bmatrix} R_i$$

Output $M = M_i$.