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Math 720: Commutative Algebra and Algebraic Geometry
Homework 2

Here is a list of problems to do. Some we will discuss in class. Those marked in bold **Turn in** should be turned in on Friday 2/10. Read through these problems. Some are things we have covered in some fashion. Some are straightforward computations, but are also good to do.

- II.2 #1-7, 10- 12
- II.3 # 5, 6-9
- II.4 #9-12
- II.5 #3, 7, 8, 13-18
- II.6 #10-12

Problem 1: TURN In: Monomial orderings (See II.4 #10-12). Each of these have fairly brief answers:

(a) For $n = 3$, list the monomials of total degree 2 or less in increasing order for grevlex and for grlex. Illustrate: for each, show a triangle with the monomials of degree 2 and the path of increasing order.

(b) Let $n = 3$. For each of the monomial orderings lex, grlex and grevlex, find vectors u_1, u_2, u_3 which determine that ordering.

(c) Find a total ordering on $\mathbb{N}_0^3$ which is determined by a single vector u_1. Explain why the one vector is sufficient.

(d) Find necessary and sufficient condition(s) on the u_i to ensure that a group ordering on $\mathbb{N}_0^n$ is also a well-ordering and therefore a monomial ordering.

(e) Find necessary and sufficient condition(s) on the u_i that ensure that a monomial ordering on $\mathbb{N}_0^n$ has no element α that is larger than an infinite number of other elements of $\mathbb{N}_0^n$. Justify your answer.

(f) Let $>_1$ be a monomial ordering on $\mathbb{N}_0^{m_1}$ and let $>_2$ be a monomial ordering on $\mathbb{N}_0^{m_2}$. Generalize the discussion in IVA II.4#10 to define the product order of $>_1$ and $>_2$. Explain how to obtain vectors defining the product order from the vectors defining the individual orders.

Problem 2: **TURN In:** (II.3 #5) Division and Gröbner basis. Use grlex with $x > y > z$ in this problem.

(a) Let $f_1 = x^2y - z$ and $f_2 = xy - 1$. Do the same analysis we did in class for division of $f = x^3 - x^2y - x^2z + x$ by f_1, f_2 and by f_2, f_1.

(b) You should get different remainders. Consider the difference of the remainders to find an element of $\langle f_1, f_2 \rangle$.

(c) Find a Groebner basis for $\langle f_1, f_2 \rangle$. It takes two steps. Prove it is a Gröbner basis using II.9 Theorem 3.

(d) Write matrix transformations that transform the original into generating set into a Gröbner basis.

(e) See if you can interpret the result geometrically.

(f) Identify a basis for $k[x, y, z]/\langle f_1, f_2 \rangle$

Problem 3: The ascending chain condition (see IVA II.5 #12-14).
(a) For a ring R prove that the following two conditions are equivalent: (i) Every ascending chain of ideals in R stabilizes. (ii) Every ideal of R is finitely generated.
(b) Show that every descending chain of varieties in k^n stabilizes.
(c) Give an example of an infinite *strictly descending* chain of ideals in $k[x]$.