## 13 The Spectrum of a Ring

This section presents a more general view of algebraic geometry. We will work over a Noetherian ring $R$ (so ideals are finitely generated), although everything here generalizes with a bit of care to arbitrary rings. As always, $k$ is a field.

### Our "geometric" object: The spectrum of a ring

**Definition 13.1.** Let $R$ be a ring.

$$\operatorname{Spec} R = \{\text{prime ideals in } R\}$$
$$\operatorname{MSpec} R = \{\text{maximal ideals in } R\}$$

*Example* 13.2. Consider the following examples

- $\mathbb{Z}$
- $D^{-1}\mathbb{Z}$ where $D$ is a multiplicatively closed set. Take $D = \left\{2^i 3^j : i, j \geq 0\right\}$ as a specific example.
- $k$
- $\mathbb{C}[x]$
- $\mathbb{R}[x]$
- $\overline{\mathbb{F}}_p[x]$
- $\mathbb{F}_p[x]$
- $\mathbb{C}[x, y]$

### The Zariski topology

We now define a topology on $\operatorname{Spec} R$. That is, a set $\mathcal{A}$ that contains subsets of $\operatorname{Spec} R$, which are called open sets, satisfying

(1) $\operatorname{Spec} R$ and $\emptyset$ are in $\mathcal{A}$.

(2) The intersection of two open sets is open.

(3) The union of any collection of open sets is open.

The closed sets of the topology are just the complements, in $\operatorname{Spec} R$, of the open sets: $\mathcal{C} = \{A^c : A \in \mathcal{A}\}$.

**Definition 13.3.** The basic open and closed sets are defined for each $f \in R$.

$$A_f = \{P : f \notin P\}$$
$$C_f = \{P : f \in P\}$$

We define $\mathcal{A}$, the open sets of $\operatorname{Spec} R$, to be arbitrary unions of the basic open sets $A_f$.

**Exercises 13.4.** The topology on $\operatorname{Spec} R$.

**(a)** Show that $A_{fg} = A_f \cap A_g$ and $C_{fg} = C_f \cup C_g$

**(b)** Show that in $k[x]$, $A_f \cap A_g = A_{\mathrm{lcm}(f,g)}$ and $A_f \cup A_g = A_{\mathrm{gcd}(f,g)}$. (It may be easier to think about the corresponding claims for the closed sets.)

**(c)** Give an example of a closed set that is not $C_f$ for some $f \in R$ (and of an open set that is not $A_f$).

**(d)** Show that $\mathcal{A}$ does satisfy the requirements to be a topology.

**Exercises 13.5.** Examples and a more general definition for open anc closed sets.

**(a)** Identify the open sets in the examples in 13.2.

**(b)** For an ideal $I$ we could define $C_I = \{P : I \subseteq P\}$. Show that if $I = \langle f_1, \ldots, f_s \rangle$ then $C_I = \bigcap_{i=1}^{s} C_{f_i}$.

**(c)** If $J = \bigcap_{s \in S} I_s$ then $C_J = \bigcup_{s \in S} C_{I_s}$.

Ring homomorphisms

Recall the following propositions about prime, radical, and maximal ideals.

**Proposition 13.6.** *Let $\varphi : R \longrightarrow S$ be a ring homomorphism.*

- *If $J$ is a radical ideal in $S$ then $\varphi^{-1}(J)$ is a radical ideal in $R$.*
- *If $J$ is a prime ideal in $S$ then $\varphi^{-1}(J)$ is a prime ideal in $R$.*
- *If $M$ is a maximal ideal in $S$ and $\varphi$ is surjective then $\varphi^{-1}(M)$ is a maximal ideal in $R$.*

Consequently, $\varphi : R \longrightarrow S$ defines a function

$$\begin{aligned} \operatorname{Spec} R &\xleftarrow{\tilde{\varphi}} \operatorname{Spec} S \\ \varphi^{-1}(Q) &\longleftarrow Q \end{aligned}$$

**Exercises 13.7.** Analyze these examples

**(a)** The natural map $k[x] \longrightarrow k[x, y]$ taking $x$ to $x$.
**(b)** The natural map $k[x] \longrightarrow k[x, y, z]$ taking $x$ to $x$ and $y$ to $y$.
**(c)** The inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$.
**(d)** The inclusion $\mathbb{R}[x] \longrightarrow \mathbb{C}[x]$.
**(e)** The map $k[x, y] \longrightarrow k[x]$ taking $x$ to $x$ and $y$ to 1.
**(f)** The map $k[x, y, z] \longrightarrow k[x]$ taking $x$ to $x$ and $y$ to $y - x$ and $z$ to 1.
**(g)** Analyze the two maps and the composite map

$$k[x, y] \longrightarrow k[x, y]/\langle y^2 - x^3 \rangle \longrightarrow k[t]$$

Where the second map is $y \longrightarrow t^3$ and $x \longrightarrow t^2$.
**(h)** Analyze the two maps and the composite map

$$k[x, y] \longrightarrow k[x, y]/\langle y^2 - x^2 + x^3 \rangle \longrightarrow k[t]$$

Where the second map is $x \longrightarrow 1 - t^2$ and $y \longrightarrow t - t^3$.
(See [CLO] §1.3#8 for parametrization of this curve.)

**Exercises 13.8.**
**(a)** Show that $\tilde{\varphi}$ is continuous.