Test 1 prep

Test 1 is March 7

In Chapter 2, we introduced term orders and I summarized and sketched the proof of the theorem of Robbiano that characterizes term orders. Once a term order is chosen, each polynomial has a unique leading term and a division algorithm is well defined. The main result so far is the Hilbert Basis Theorem (HBT), which says that any ideal in $k[x_1, \ldots, x_n]$ is finitely generated. For a specified term order, we can define a special type of generating set, called a Groebner basis, that give us a good way of describing ideals, determining ideal membership and computing. Buchberger's theorem tells us when a generating set is a Groebner basis and Buchberger's algorithm derives a Groeber basis from an arbitrary generating set.

(1) A term (or monomial) order is an ordering on the monomials (or equivalently, on $\mathbb{N}^n$) that is a total order, a well order, and a semigroup order (it respects $+$).

- Robianno proved that any term order is characterized by a sequence of vectors $u_1, \ldots, u^r \in \mathbb{R}^n$: $\alpha < \beta$ if and only if there is some t such that $\alpha \cdot u_i = \beta \cdot u_i$ for $i < t$ but $\alpha \cdot u_t < \beta \cdot u_t$.
- Suppose $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ each determine a term order. What conditions on the u_i and v_i ensure that the term orders they define are the same?

(2) Given a term ordering we can define a division algorithm in $k[x_1, \ldots, x_n]$.

- Given a polynomial h and polynomials $f_1, \ldots, f_s$ the division algorithm produces quotients q_i and a remainder r such that

$$h = f_1 q_1 + \cdots + f_s q_s + r, \text{ and}$$
$$r \text{ has no term that is divisible by } \mathrm{LT}(f_i) \text{ for any } i \in \{1, \ldots, s\}$$

- Reordering the polynomials can give different quotients and remainders. The difference of two remainders resulting from different orderings is in the ideal generated by $f_1, \ldots, f_s$.

(3) A ring R is called **Noetherian** when it satisfies the ascending chain condition (ACC), that is, any increasing chain of ideals stabilizes. The ACC is true if and only if every ideal in R is finitely generated.

- Hilbert's Basis Theorem says that if a ring R satisfies the ACC then so does $R[y]$. Consequently, every ideal in $k[x_1, \ldots, x_n]$ is finitely generated.
- Dickson's lemma says something a little bit stronger, but only applying to monomial ideals: Given any subset $S \subseteq \mathbb{N}^n$ there is a *finite* subset $T \subseteq S$ such that $\langle x^\beta : \beta \in T \rangle = \langle x^\beta : \beta \in S \rangle$. We can choose our finite generating set from the given generators.

(4) Fix a term order on $k[x_1, \dots, x_n]$. Given an ideal I, the ideal of leading terms $\langle LM(I) \rangle = \langle LM(f) : f \in I \rangle$ (a monomial ideal) is finitely generated, by Dickson's Lemma (or HBT). Let $x^{\alpha_1}, \dots, x^{\alpha_s}$ generate it. We may assume this set is minimal: no x^{α_i} divides x^{α_j} for $j \neq i$.

- Let $g \in I$ have $LT(g_i) = x^{\alpha_i}$. The ideal I is generated by the polynomials g_i. This is a minimal Groebner basis. Replacing each g_i by its remainder upon division by the others gives a reduced Groebner basis.
- The **footprint** of I is $\Delta_I = \mathbb{N}_0^n \setminus \{\mathrm{LE}(f) : f \in I\}$.
- The set $\left\{ x^\beta : \beta \in \Delta_I \right\}$ is a basis for $k[x_1, \dots, x_n]/I$.
- Computation in $k[x_1, \dots, x_n]/I$ is done by reducing modulo the Groebner basis.

(5) Buchberger's theorem says that $G = \{g_1, \dots, g_s\}$ is a GB for the ideal it generates, $\langle G \rangle$, iff each syzygy polynomial $S(g_i, g_j)$ reduces to 0 when divided by G.

- Given an arbitrary generating set $F = F_0$ we may construct a GB using Buchberger's algorithm. The algorithm proceeds iteratively. Roughly speaking, we can identify of two steps:
  (a) Divide each $f \in F$ by $F \setminus \{f\}$ to eliminate any redundancy.
  (b) Compute a syzygy polynomial of two polynomials in the generating set $S(f, h)$, then divide $S(f, h)$ by the generating set to get a remainder. If the remainder r is nonzero, it is added to the generating set to get $F_1 = F_0 \cup \{r\}$. The algorithm proceeds with this enlarged generating set.
- If $\mathrm{GCD}\big(\mathrm{LM}(f), \mathrm{LM}(h)\big) = 1$ then the syzygy polynomial of f, h does reduce to 0 (by an exercise), so there is no need to compute and test it.
- In Buchberger's algorithm the leading monomials of the F_i generate an increasing sequence of monomial ideals. By the HBT this increasing sequence stabilizes, consequently the algorithm must terminate in a finite number of steps.
- (An improvement) A careful look at the proof of Buchberger's theorem shows that it is only necessary to assume that $S(g_i, g_j)$ has an LCM representation, that is

$$S(g_i, g_j) = \sum_{k=1}^{s} h_k g_k \qquad \text{with } \mathrm{LM}(h_i g_i) < \mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))$$

to ensure that $\{g_1, \dots, g_s\}$ is a Groebner basis.

Discussion questions and Homework 4, 5

Discussion items for 2/16 (B) (C) (D2,4)
Turn in (A) and (D1) on Tuesday 2/21.

(A) [OS] 7.3 on monomial orderings.

(B) The ascending chain condition (see [CLO] 2.5 #12-14).

(1) For a ring R prove that the following two conditions are equivalent: (1) Every ascending chain of ideals in R stabilizes. (2) Every ideal of R is finitely generated.

(2) Show that every descending chain of varieties in k^n stabilizes.

(3) Give an example of an infinite *strictly descending* chain of ideals in $k[x]$.

(C) Properties of Groebner Bases

(1) [CLO] 2.6 #13; 2.7 #5-7

(2) [CLO] 2.6 # 1, 3, 4, uniqueness of remainder.

(D) Examples of Groebner Bases

(1) (HW) [CLO] §2.3#5a-c Reordering gives different remainders.

(2) [CLO] 2.3#9, 10; 2.5 #8; 2.6 #9, 10ab. Implicitization of a twisted rational curve and similar problems.

(3) [CLO] 2.5 #7; 2.6 #9c, 10; 2.7 #2, 3. computation of GB.

(4) [CLO] 2.5 #9, 2.7 #9,10 linear polynomials and GB.

(5) [CLO] 2.7#14 Lagrange interpolation of points in the plane.

(E) Syzygy polynomials

(1) [CLO] 2.6#5, 6,7 compute the syzygy polynomial.

(2) [CLO] 2.6#7, 8, 12 syzygy and monomial multiples.

Discussion for 2/21

Turn in (F2,5) and (G) on Tuesday 2/28.

(F) Buchberger and Groebner bases

(1) [CLO] 2.8 #1, 2, 6, Groebner basis computations,

(2) (HW) [CLO] 2.8 #7 implicitization of parametrically defined surface.

(3) [CLO] 2.9 #1, 2 standard representation and Groebner bases.

(4) [CLO] 2.10 #1, syzygy and module properties.

(5) (HW) [CLO] 2.10 # 2, 3 syzygy and determinental ideals.

(G) (HW) Groebner basis and computations. In $k[x, y, z]$ with the grlex term order with $x > y > z$, let $I = \langle g_1, g_2, g_3, g_4 \rangle$ where,

$$g_1 = x^2 - x \qquad g_2 = xy - z$$
$$g_3 = xz - z \qquad g_4 = yz - z^2$$

(1) Show that $S(g_1, g_2)$ and $S(g_3, g_4)$ reduce to 0 when divided by $G = [g_1, g_2, g_3, g_4]$. If you did this for all pairs g_i, g_j you would find that this is a Groebner basis for I (trust me).

(2) Identify a basis for $k[x, y, z]/I$ using this Groebner basis. It may help to graph the leading terms of G.

(3) Write down a general element of $k[x, y, z]/I$.

(4) Explain how to compute in $k[x, y, z]/I$. In particular, show that $y^i z^j = z^{i+j}$.

(5) Show that the associated variety is the union of two lines defined by ideals I_1 and I_2. Show that I_1 contains I and similarly for I_2.

(6) Analyze the ring map

$$k[x, y, z]/I \longrightarrow k[x, y, z]/I_1 \times k[x, y, z]/I_2$$

Is it injective? Is it surjective?