1 Ring Homomorphisms

Let's recall some results about ring homomorphisms. (These apply equally to K-algebra homomorphisms as a special case).

Recall Theorem 5.5 from week 2, that associates a radical/prime/maximal ideal I with the ring R/I being a reduced ring/integral domain/field.

From a week 4 assignment,

Proposition 1.1. *If $\varphi : R \longrightarrow S$ is a homomorphism and $I < S$ is radical/prime then $\varphi^{-1}(I)$ is radical/prime. When φ is surjective and I maximal $\varphi^{-1}(I)$ is maximal.*

Theorem 1.2 (Factor/First Isomorphism)**.** *Let $\varphi : R \longrightarrow S$ be a ring homomorphism with kernel K. Then φ factors as a composition of a surjective homomorphism followed by an injective homomorphism. When φ is surjective, $S \cong R/K$.*

Theorem 1.3 (Factor/First Isomorphism)**.** *Let $\varphi : R \longrightarrow S$ be a surjective ring homomorphism with kernel K (so $S \cong R/K$). There is a 1-1 correspondence between ideals of S (or R/K) and ideals of R containing K. For $I \supseteq K$, we have $R/I \cong R/K \Big/ I/K$.*

Furthermore (by week 2 Thm 5.5), the correspondence respects our standard properties: radical/prime/maximal ideals correspond to the same.

Observations:

(1) Surjective homomorphisms are "easy" to understand: they are just modding out by an ideal

(2) Injective homomorphisms seem manageable (at first glance): if $\varphi : R \longrightarrow S$ is injective, $\varphi(R)$ is a subring of S and is isomorphic to R, so how hard can that be?

(3) Any homomorphism is a composite of an injective homomorphism followed by a surjective homomorphism, so if we understand each type, we understand all homomorphisms.

2 Coordinate Rings and Function Fields

Let $V \subseteq \mathbb{A}_K^m$ be a variety. Consider elements of $K[x_1, \ldots, x_m]$ as functions on $\mathbb{A}_K^m$, which can be restricted to V.

Definition 2.1. The **coordinate ring** of V is $K[V] = K[x_1, \ldots, x_m]/\mathbb{I}(V)$. It is the **ring of polynomial functions** on V.

We will use $\tilde{x}_i$ for the coset $x_i + \mathbb{I}(V)$ in $K[x_1, \ldots, x_m]/\mathbb{I}(V)$ and similarly for $f \in K[x_1, \ldots, x_m]$, we write $\tilde{f}$ for the image in the quotient ring.

Definition 2.2. A K-algebra R is **finitely-generated** when there are elements $a_1, \ldots, a_m$ in R such that every element of R can be expressed as a polynomial in the a_i with coefficients from K.

It is pretty clear to see that

Proposition 2.3. *A K-algebra R is finitely-generated (by $a_1, \ldots, a_m$) if and only if there is a surjective homomorphism $K[x_1, \ldots, x_m]$ to R (that takes x_i to a_i).*

The coordinate ring of an algebraic variety over K is a finitely-generated K-algebra and it is also reduced.

Note that the coordinate ring of an irreducible variety will be an integral domain since the ideal of an irreducible variety is prime. We will see later that if K is algebraically closed, the converse of the last point is true: if R is finitely-generated over K and reduced then it is isomorphic to the coordinate ring of a variety over K.

Example 2.4. Over a finite field $\mathbb{F}_q$ there are only a finite number of points, so for example, $x^q - x$ vanishes on all points of $\mathbb{A}^1_{\mathbb{F}_q}$. Thus $K[\mathbb{A}^1_{\mathbb{F}_q}] = K[x]/\langle x^q - x\rangle$.

Over an infinite field, $K[\mathbb{A}^m_{\mathbb{F}_q}] = K[x_1, \ldots, x_m]$. This is proven by induction with the base step being that a polynomial in $K[x]$ can have only a finite number of roots.

When we study dimension we will use the following object.

Definition 2.5. Let V be an irreducible variety and let $K[V]$ be its coordinate ring. The field of fractions of $K[V]$ is called the **function field** of V and denoted $K(V)$. The elements are called **rational functions** on V.

3 Geometry and Surjective Homomorphisms

Now let's interpret the relationship between ring homomorphisms and maps (polynomial functions) from one variety to another. We'll focus on subvarieties of $\mathbb{A}^m$ and associated surjective homomorphisms from $K[x_1, \ldots, x_m]$.

For $\bar{p} \in \mathbb{A}^m_K$, we can think of the inclusion of $\bar{p}$ into $\mathbb{A}^m$ as a polynomial map from $\mathbb{A}^0$ to $\mathbb{A}^m$. There is an associated surjective homomorphism.

$$\mathbb{I}(\bar{p}) \text{ is the maximal ideal } \langle x_1 - p_1, \ldots, x_m - p_m \rangle$$
$$K[\bar{p}] = \frac{K[x_1, \ldots, x_m]}{\langle x_1 - p_1, \ldots, x_m - p_m \rangle} \cong K$$

For a variety $\mathbb{V} \subseteq \mathbb{A}^m_K$ we have the following equivalences,

$$\begin{aligned} f \in \mathbb{I}(V) &\iff f(\bar{p}) = 0 \text{ for all } \bar{p} \in V \\ &\iff f \in \mathbb{I}(\bar{p}) \text{ for all } \bar{p} \in V \\ &\iff f \in \bigcap_{\bar{p} \in V} \mathbb{I}(\bar{p}) \end{aligned}$$

Consequently,

$$\mathbb{I}(V) = \bigcap_{\bar{p} \in V} \mathbb{I}(\bar{p})$$

Example 3.1. Over a finite field $\mathbb{F}_q$ there are only a finite number of points, so for example, $x^q - x$ vanishes on all points of $\mathbb{A}^1_{\mathbb{F}_q}$. Thus $K[\mathbb{A}^1_{\mathbb{F}_q}] = K[x]/\langle x^q - x \rangle$.

Over an infinite field, $K[\mathbb{A}^m_{\mathbb{F}_q}] = K[x_1, \ldots, x_m]$. This is proven by induction with the base step being that a polynomial in $K[x]$ can have only a finite number of roots.

Notice the following things to add to our AG-dictionary:

- $K[V]$ is a reduced ring, since $\mathbb{I}(V)$ is radical. (More generally, for an arbitrary I, $K[x_1, \ldots, x_m]/I$ may have nilpotents.)
- $K[V]$ will be an integral domain iff V is an irreducible variety (since both are equivalent to $\mathbb{I}(V)$ being prime).
- $K[\bar{p}]$ is a maximal ideal for $\bar{p}$ a point in $\mathbb{A}^m_K$, namely $\langle x_1 - p_1, \ldots, x_m - p_m \rangle$. Furthermore, $K[\bar{p}] = K[x_1, \ldots, x_m]/\langle x_1 - p_1, \ldots, x_m - p_m \rangle \cong K$.

For an arbitrary maximal ideal M we know that $K[x_1, \ldots, x_m]/M$ is a field. Proving it is a finite degree extension (that is, isomorphic to $K[x]$ modulo an irreducible polynomial) is a key part of the Nullstellensatz.

Now consider two varieties $\mathbb{A}^m_K \supseteq V \supseteq W$. Think of this indicating functions (inclusion) from W to V and V to $\mathbb{A}^m_K$. I claim that there are natural surjective homomorphisms (which go, of course, in the other direction)

$$K[\mathbb{A}^m_K] \longrightarrow K[V] \longrightarrow K[W]$$

This comes from the 3rd isomorphism theorem. Since $V \supseteq W$, $\mathbb{I}(V) \subseteq \mathbb{I}(W)$. Therefore, $K[V] = K[x_1, \ldots, x_m]/\mathbb{I}(V)$ has an ideal $\mathbb{I}(W)/\mathbb{I}(V)$ and

$$K[V] \longrightarrow \Big(K[x_1, \ldots, x_m]/\mathbb{I}(V)\Big)\Big/\Big(\mathbb{I}(W)/\mathbb{I}(V)\Big) \cong K[x_1, \ldots, x_m]/\mathbb{I}(W) = K[W]$$

In particular, for $V \ni \bar{p}$, $K[V] \longrightarrow K[\bar{p}]$

$$K[V] \longrightarrow K[x_1, \ldots, x_m]/\mathbb{I}(\bar{p}) = K[x_1, \ldots, x_m]/\langle x_1 - p_1, \ldots, x_m - p_m\rangle \cong K$$

4 Geometry and Injective Homomorphisms

Injective homomorphisms are not as simple as one might hope.

Example 4.1. $K[x, xy, xy^2, xy^3, \ldots]$ is a subring of $K[x, y]$ but it is not Noetherian.

We will avoid such problems by only dealing with finitely-generated rings. Nevertheless, things are varied and complicated enough that we should begin with several examples. We will proceed as follows.

(1) Injective maps $K[y_1, \ldots, y_n]$ to $K[x_1, \ldots, x_m]$.

(2) Injective maps taking a subring $K[f_1(\bar{x}), \ldots, f_2(\bar{x})]$ to $K[x_1, \ldots, x_m]$.

(3) Arbitrary maps $K[y_1, \ldots, y_m]$ to $K[x_1, \ldots, x_m]$.

(4) Arbitrary maps $K[V]$ to $K[W]$ for $W \subseteq \mathbb{A}^m$ and $V \subseteq \mathbb{A}^m$ two varieties.

Example 4.2. For $n \leq m$, consider $\varphi : K[y_1, \ldots, y_n] \longrightarrow K[x_1, \ldots, x_m]$ with $\varphi(y_i) = x_i$. Using φ, each point $\bar{p} \in \mathbb{A}^m$ gives a well-defined point of $\mathbb{A}^m$. We have

$$K[\bar{y}] \longrightarrow K[\bar{x}] \longrightarrow K[\bar{x}]/\langle x_1 - p_1, \ldots, x_m - p_m\rangle \cong K$$

The composite of these maps is surjective(!), since elements of K in $K[\bar{x}]$ have to go to themselves. So the preimage of $0 \in K[\bar{p}]$ (equivalently of $\langle y_1 - p_1, \ldots, y_m - p_m\rangle \in K[\bar{y}]$) is some maximal ideal in $K[\bar{x}]$. But we can see that the composite map takes y_i to p_i, so that maximal ideal in $K[\bar{y}]$ is $\langle y_1 - p_1, \ldots, y_m - p_m\rangle$. Geometrically, we have a map from $\mathbb{A}^m$ to $\mathbb{A}^n$ taking $(p_1, \ldots, p_m)$ to $(p_1, \ldots, p_n)$. Check!

If we tweaked the map a bit by taking y_i to $x_i + c_i$ (for $c_i \in K$) then the corresponding map of varieties would be $(p_1, \ldots, p_m)$ to $(p_1 + c_1, \ldots, p_m + c_m)$

Example 4.3.

$$\begin{aligned} K[y_1, y_2] &\longrightarrow K[x_1, x_2, x_3] \\ y_1 &\longmapsto x_1 + x_2 \\ y_2 &\longmapsto x_2 + 2x_3 \end{aligned}$$

Example 4.4. $K[x] \longrightarrow K[x, y]/\langle y - x^2\rangle$

Example 4.5. $K[y] \longrightarrow K[x, y]/\langle y - x^2\rangle$

Example 4.6. $K[x] \longrightarrow K[x, y]/\langle xy - 1\rangle$

5 Maps of varieties and homomorphisms of rings

Here are some comments on [CR: Ch 3-4]. To keep track of where we are working I will use the following notation (which differs a bit from theirs). We are working over some specific field K .

$\mathbb{A}^m$	$\mathbb{A}^n$
ring $K[x_1, \ldots, x_m]$	ring $K[y_1, \ldots, y_n]$
polynomials f	polynomials g
variety V	variety W
point $\bar{p}$	point $\bar{q}$

A polynomial map F from a variety V in $\mathbb{A}_K^m$ to a variety W in $\mathbb{A}_K^n$ is a function given by n polynomials in the indeterminates $x_1, \ldots, x_m$. That is, it is determined by

$$f_1(x_1, \ldots, x_m), f_2(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m) \in K[x_1, \ldots, x_m]$$

BUT, we must check that the image of V lies in W . Observe:

- Each $f_i(x_1, \ldots, x_m)$ determines a function from $\mathbb{A}^m$ to $\mathbb{A}^1$ that is polynomial.
- Any choice of n of these functions in $K[x_1, \ldots, x_m]$ determines a function to $\mathbb{A}^n$.
- For a point $\bar{p} \in \mathbb{A}^m$, the image is $\Big(f_1(p_1, \ldots, p_m), \ldots, f_n(p_1, \ldots, p_m)\Big)$.
- The functions can be restricted to V to give a function $V \longrightarrow \mathbb{A}^n$.
- It is not generally true that the image lies in W . This must be verified:
  - Is it true that for all $g \in \mathbb{I}(W)$ and $\bar{p} \in V$ it holds that $g(F(\bar{p})) = 0$?

Interpreting this algebraically we have

$$\begin{aligned} K[\bar{p}] \longleftarrow \frac{K[x_1, \ldots, x_m]}{\mathbb{I}(V)} \xleftarrow{\pi} K[x_1, \ldots, x_m] &\xleftarrow{\varphi} K[y_1, \ldots, y_n] \\ f_i(x_1, \ldots, x_m) &\longleftarrow y_i \\ g \circ f &\longleftarrow g(y_1, \ldots, y_n) \end{aligned}$$

where $g \circ f = g\big(f_1(x_1, \ldots, x_m), \ldots, f_i(x_1, \ldots, x_m)\big)$.

To get a function $V \longrightarrow W$ we must check that for all $g \in \mathbb{I}(W)$ and $\bar{p} \in V$ it holds that $g(F(\bar{p})) = 0$. We have these equivalences

$$\begin{aligned} g(F(\bar{p})) = 0 &\Longleftrightarrow g\big(f_1(p_1, \ldots, p_m), \ldots, f_i(p_1, \ldots, p_m)\big) \\ &\Longleftrightarrow \varphi(g) \in \mathbb{I}(\bar{p}) \end{aligned}$$

Thus we must check that $\varphi\big(\mathbb{I}(W)\big) \subseteq \mathbb{I}(V)$.

Proposition 5.1 (CR: 4.17)**.** *Let V be a variety in $\mathbb{A}^m$ and W a variety in $\mathbb{A}^n$. There is a bijection between polynomial maps $V \longrightarrow W$ and K-algebra homomorphisms $K[V] \longleftarrow K[W]$.*

Under this correspondence (1) composition of polynomial maps corresponds to composition of k-algebra homomorphisms (in the reverse direction) and (2) an isomorphism of rings corresponds to an isomorphism of varieties.

6 Problems

Problems 6.1. Geometry of some ring embeddings

For each of the following polynomials $f(x,y)$, explain the geometry for each of the embeddings $\mathbb{R}[x] \longrightarrow \mathbb{R}[x,y]/f$, $\mathbb{R}[y] \longrightarrow \mathbb{R}[x,y]/f$, $\mathbb{C}[x] \longrightarrow \mathbb{C}[x,y]/f$, $\mathbb{C}[y] \longrightarrow \mathbb{C}[x,y]/f$. Which points of $\mathbb{A}^1$ are in the image? Which points of $\mathbb{A}^1$ have 1, 2, or 3 preimages? Can you sketch?

(a) $f = y^2 - x^3$

(b) $f = y^2 - x^3 + 1$

(c) $f = y^2 - x^3 + x$

(d) $f = y^2 - x^3 + x^2$

(e) $f = y^2 - x^3 - x^2$

Problems 6.2. Geometry of an affine map

For $m < n$, consider $\varphi : K[y_1, \ldots, y_n] \longrightarrow K[x_1, \ldots, x_m]$ with $\varphi(y_i) = a_i x_i + b_i$.

(a) What is the function from $\mathbb{A}^n$ to $\mathbb{A}^m$ that corresponds to this homomorphism? (Suggestion, try $m = n = 1$ to start than gradually add to the dimensions.)

(b) Generalize to an arbitrary affine map.

Problems 6.3.

(a) Show that $\mathbb{V}(\langle x^2 - xy\rangle)$ is reducible.

(b) Create an example of an ideal $I \subseteq K[x,y]$ such that $\mathbb{V}(I)$ consists of a point union a line (and is therefore reducible).

Problems 6.4.

(a) Let R be a reduce ring. Prove that any subring of R is also reduced.

(b) Prove that $R[x]$ is reduced.

Problems 6.5. Subrings and varieties

Let $g_1, \ldots, g_n \in K[x_1, \ldots, x_m]$ and consider the homomorphism $K[y_1, \ldots, y_n] \longrightarrow K[x_1, \ldots, x_m]$ taking x_i to g_i.

(a) Show the kernel of the homomorphism must be a prime ideal, I.

(b) Let $V = \mathbb{V}(I)$. Show that $K[V] \cong K[g_1, \ldots, g_n]$.

(c) Apply to $K[u^2, uv, v^2]$. What is the kernel?

Problems 6.6.

(a) $K[x] \longleftarrow K[y]$ taking y to x^2 gives a surjective map of varieties when $K = \mathbb{C}$ but not $\mathbb{R}$.

(b) $\mathbb{C}[x] \longleftarrow \mathbb{C}[y]$ taking y to $f(x)$ is generally d to 1 when $d = \deg(f)$, but not at points p that are roots of $f'(x)$.

Problems 6.7. [CR: 4.2.7-8] Isomorphic varieties

(a) Show that $K[x,y,z,w]/\langle y - f(x), z - g(x), w - h(x)\rangle$ is isomorphic to a polynomial ring.

(b) Prove $\mathbb{V}(xy-1)$ is not isomorphic to $\mathbb{A}^1$.

(c) Prove $\mathbb{A}^1$ is not isomorphic to $\mathbb{A}^2$.

(d) Prove $\mathbb{V}(z - x^2 + xy) \subseteq \mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$

(e) Prove $\mathbb{V}(xy) \subseteq \mathbb{A}^2$ is isomorphic to $\mathbb{V}(y^2 - x^2) \subseteq \mathbb{A}^2$.

Problems 6.8. Polynomial maps and their associated ring homomorphisms
[CR: 4.2.9] Let $F : W \longrightarrow V$ be a polynomial map of varieties and let $F^* : K[W] \longrightarrow K[V]$ be the associated homomorphism of coordinate rings.

(a) Prove that if F^* is surjective then F is injective.

(b) Prove that if F is surjective then F^* is injective.

(c) Give an example in which F^* is injective but F is not surjective.

(d) Is there an example in which F is injective but F^* is not surjective?

Problems 6.9. Collapsing maps

(a) [CR: 4.3.6] Let $V \subseteq \mathbb{A}^m$ be an affine variety and let $W \subseteq \mathbb{A}^n$ be a single point, $W = \bar{q}$. There is only one possible map $V \longrightarrow \bar{q}$. Find the associated map of polynomial rings $K[\bar{q}] \longrightarrow K[V]$.

(b) [CR: 4.3.7] Suppose that $V \subseteq \mathbb{A}^m$ is irreducible let $W \subseteq \mathbb{A}^n$ be consist of two points. Show there are exactly two possible maps $V \longrightarrow \bar{q}$. Write down the associated maps of polynomial rings $K[W] \longrightarrow K[V]$.