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Math 627A: Modern Algebra I

Homework III

Problem 8 For an abelian group A and integer m we let $mA = \{ma : a \in A\}$. Verify for yourself that this is a group. Let p be a prime number.
(a) Show that $p^a\mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^{n-a}}$ for $a \leq n$. (b) Letting $n = a+b$ in part (a), show that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^b} \xrightarrow{\cdot p^a} \mathbb{Z}_{p^{a+b}} \longrightarrow \mathbb{Z}_{p^a} \longrightarrow 0$$

(b) Show that $p^{a-1}\mathbb{Z}_{p^n} \Big/ p^a\mathbb{Z}_{p^n} \cong \mathbb{Z}_p$ for $a \leq n$.
(c) Suppose that $A \cong (\mathbb{Z}_p)^{a_1} \times (\mathbb{Z}_{p^2})^{a_2} \times \cdot \times (\mathbb{Z}_{p^n})^{a_n}$. Show that $p^{t-1}A/p^tA \cong (\mathbb{Z}_p)^{a_t+\cdots+a_n}$.
(d) Conclude the uniqueness part of the classification of finite abelian groups: If

$$(\mathbb{Z}_p)^{a_1} \times (\mathbb{Z}_{p^2})^{a_2} \times \cdot \times (\mathbb{Z}_{p^n})^{a_n} \cong (\mathbb{Z}_p)^{b_1} \times (\mathbb{Z}_{p^2})^{b_2} \times \cdot \times (\mathbb{Z}_{p^n})^{b_n}$$

then $a_i = b_i$. (The a_i and b_i may be 0 in the isomorphism.)

Problem 9 [H 7.8 #17] Let G be the group of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

with $a, b, c \in \mathbb{Q}$.
(a) Find the center C of G and show that C is isomorphic to the additive group $\mathbb{Q}$.
(b) Show that the center of G/C is isomorphic to $\mathbb{Q} \times \mathbb{Q}$.
(c) Conclude that G is metabelian.

Problem 10 Let p, q and r be prime and let $n = p^6q^2r^4$.
(a) How many abelian groups are there of order n (up to isomorphism)?
(b) For each i from 1 to 7 find how many of these groups have exactly i invariant factors?