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# Math 627B: Modern Algebra II

## Homework III

**Problem 1:** Monomial orderings:
(a). For $n =$, list the monomials of total degree 2 or less in increasing order for grevlex and for grlex. This is enough to show they are different orderings.
(b). Let $n = 3$. For each of the monomial orderings lex, grlex and grevlex, find vectors $u_1, u_2, u_3$ which determine that ordering.
(c) Find a total ordering on $\mathbb{N}_0^3$ which is determined by a single vector $u_1$. Explain why the one vector is sufficient.
(d) Find necessary and sufficient condition(s) on the $u_i$ to ensure that a group ordering on $\mathbb{N}_0^n$ is also a well-ordering and therefore a monomial ordering.
(e) Find necessary and sufficient condition(s) on the $u_i$ that ensure that a monomial ordering on $\mathbb{N}_0^n$ has no element $\alpha$ that is larger than an infinite number of other elements of $\mathbb{N}_0^n$. Justify your answer.
(f) Let $>_1$ be a monomial ordering on $\mathbb{N}_0^{m_1}$ and let $>_2$ be a monomial ordering on $\mathbb{N}_0^{m_2}$. Generalize the discussion in IVA II.4#10 to define the product order of $>_1$ and $>_2$.
Explain how to obtain vectors defining the product order from the vectors defining the individual orders.

**Problem 2:** Division: IVA II.3#9

**Problem 3:** The ascending chain condition (see IVA II.5 #12-14).
(a) For a ring $R$ prove that the following two conditions are equivalent: (i) Every ascending chain of ideals in $R$ stabilizes. (ii) Every ideal of $R$ is finitely generated.
(b) Show that every descending chain of varieties in $k^n$ stabilizes.
(c) Give an example of an infinite *strictly descending* chain of ideals in $k[x]$.

**Problem 4:** Groebner bases: Fix a monomial ordering $<$.
(a) Show that the remainder of $f$ when divided by $I$ is well defined. It is independent of the Groebner basis (IVA II.6 #4).
(b) Do problems IVA II.7 #2, 3.
(c) Read problems IVA II.7#5-7.9.