Mike O'Sullivan
Department of Mathematics
San Diego State University

Fall 2012

Math 627A: Modern Algebra I
Homework I

Problem 1: Subgroup lattice diagrams.

(a) Draw the diagrams for $\mathbb{Z}_4 \times \mathbb{Z}_2$, the Quaternions, Q, and the Alternating Group A_4.

(b) Notice that the diagrams that we did in class and the ones you created establish that $\mathbb{Z}_8$, $\mathbb{Z}_2^3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_4 and Q are pairwise nonisomorphic.

(c) Use sage to compute the dicyclic group of order 12. Find as many qualities of this group as you can to distinguish it from A_4 and from D_6.

Problem 2: Automorphisms of $\mathbb{Z}_n$.

(a) Prove that $\mathrm{Aut}(\mathbb{Z}_n) \cong U_n$, the group of units in $\mathbb{Z}_n$.

(b) Each group U_n is isomorphic to a cyclic group or a direct product of such. For each of $n = 8, 9, 10, 11, 12$ find the product of cyclic groups that is isomorphic to U_n.

Problem 3: Exercise 1.20 and 2.7b on order:

(a) If $g \in G$ has order m and $h \in H$ has order n, find the order of $(g, h) \in G \times H$.

(b) Suppose that $a, b \in G$ commute (that is $ab = ba$). If $\mathrm{ord}(a)$ and $\mathrm{ord}(b)$ are coprime find the order of ab.

(c) Let A be an abelian group such that $\exp(A)$ is finite. Show that there is some $a \in A$ such that $\mathrm{ord}(a) = \exp(A)$.

(d) Show that S_4 has no element with order equal to $\exp(S_4)$. [Use intelligent brute force.]

(e) The order of $\pi \in S_n$ is the lcm of the signature list.