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Math 627A: Modern Algebra I
Homework 2

Problem 1: Problem 3.12 in my notes. Let $\pi \in S_n$. For any $\sigma \in S_n$, the signature of σ and the signature of $\pi\sigma\pi^{-1}$ are the same.

(a) Consider first the case where σ is a t-cycle and π is a transposition. Show that $\pi\sigma\pi^{-1}$ is a t-cycle. [You will have to consider 3 cases based on $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\pi)$.]

(b) Extend to arbitrary π by noting that every permutation is the product of transpositions.

(c) Extend to arbitrary σ by writing σ as the product of disjoint cycles and using the fact that conjugation by π "respects products."

Problem 2:

(a) Identify all possible signatures for elements of S_5.

(b) For each signature, count how many elements have that signature. Check that you get the correct total number of elements in S_5.

Problem 3:

(a) Show that A_n is invariant under conjugation: for any $\pi \in S_n$, $\pi A_n \pi^{-1} = A_n$. (Problem 3.13a)

(b) Let C_n be the cyclic subgroup of D_n. Find two elements of C_4 that are conjugate as elements of D_4 but are not conjugate as elements of C_4.

(c) Find two elements of D_4 that are conjugate as elements of S_4 but are not conjugate as elements of D_4. You may use sage.

Problem 4:

(a) Show that the subgroup consisting of the upper triangular matrices in $\mathrm{Gl}(2)$ is conjugate to the subgroup of lower triangular matrices in $\mathrm{Gl}(2)$. [Hint: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.]

(b) Show that the set of matrices with nonzero determinant of the form $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$ is a coset of the upper triangular matrices.

Problem 5: See Problem 3.10.

(a) Let a be an element of a group G. Show that φ_a defined by $\varphi_a(g) = aga^{-1}$ is an automorphism of G.

(b) Show that there is a homomorphism $\varphi : G \longrightarrow \mathrm{Aut}(G)$ defined by $a \longrightarrow \varphi_a$.

(c) What can you say about the kernel of φ?