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Math 627B: Modern Algebra II
Homework VI

Due Tu. 11/27, 2012.

Problem 1: Let F be a field and let G be the subgroup of $\mathrm{Gl}(F,n)$ that stabilizes the standard basis vector $\begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$

(a) Show that G has a subgroup H isomorphic to $\mathrm{Gl}(F,n-1)$.

(b) Show that G has a normal subgroup N isomorphic to the additive group F^{n-1}.

(c) Show G is the semidirect product $N \rtimes H$.

Problem 2: Nonabelian groups of order 2^n.

(a) For both the quaternions, Q, and the dihedral group with 8 elements, D_4, the center is isomorphic to $\mathbb{Z}_2$ and the quotient to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 nontrivial proper subgroups.

- Using $D_8/Z(D_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, find the subgroups of D_8 corresponding to the 3 proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Find the isomorphism class of each of these subgroups ($\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.).

- Do the same thing for Q.

(b) Let G be a non-abelian group of order 16. Identify all possibilities for Z the center of G. For each possible center identify the possibilities for G/Z. Give a short justification for your answers.

Problem 3: Let P be a p-Sylow subgroup of G. Let N be a normal subgroup of G. Show that

(a) $P \cap N$ is a p-Sylow subgroup of N.

(b) Show that PN/N is a p-Sylow subgroup of G/N.

Note: If p does not divide $|G|$, the p-Sylow subgroup of G is the trivial group, $\{e\}$.

Problem 4: In this problem we generalize the result that a group of index 2 must be normal. (You may use the results in Problems 5.1#1,2 in Ash. See also 5.1 #8, 9; and 5.5#8,9.)
Let H be a subgroup of a finite group G of index n, so $[G : H] = n$.

(a) Let G act on left cosets of H by left multiplication. Let N be the kernel of the action. Show that $[G : N]$ divides $n!$.

(b) Suppose in addition that $n = p$ is prime. Show that $[G : N]$ divides $(p-1)!$.

(c) Suppose in addition that p is the smallest prime dividing $|G|$. Show that H is normal in G.