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# Math 627B: Modern Algebra II
## Review for First Exam

Please read the following problems and their solutions in Ash's text. Many of them are routine, and others have been covered in class to some extent. Ash also provides solutions. (**Note**: Ash writes $D_n$ as $D_{2n}$.)

- §1.1 pr. 2-11.
- §1.2 pr. 1-8.
- §1.3 pr. 1-12.
- §1.4 pr. 1-9.
- §1.5 pr. 1-8.
- §5.1 pr. 1, 2, 4, 5, 8, 9
- §5.2 pr. 2-5

Let $G$ be a group. The following groups and subgroups are important. You should be able to establish these results.

- $Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$ is a normal subgroup of $G$.
- The centralizer of $a \in G$, $C(a) = \{g \in g : ga = ag\}$ is a subgroup of $G$ containing $a$. $Z(G) = \bigcap_{a \in A} C(a)$.
- Let $H$ be a subgroup of $G$. The normalizer of $H$, $N_H = \{x \in G : x^{-1}Hx = H\}$ is a subgroup of $H$ containing $H$. $H$ is normal in $N_H$. Any subgroup $K$ of $G$ that contains $H$ as a normal subgroup is contained in $N_H$. (If $N \trianglelefteq K \leq G$ then $K \leq N_H$.)
- The set of automorphisms of $G$, $\operatorname{Aut}(G)$, is a group.
- The set of inner automorphisms of $G$, $\operatorname{Inn}(G)$ is a normal subgroup of $\operatorname{Aut}(G)$.
- If $G$ is *abelian*, $\operatorname{Tor}(G) = \{a \in G : \operatorname{ord}(a) \text{ is finite }\}$ is a normal subgroup of $G$ and $G/\operatorname{Tor}(G)$ has no elements of finite order.

**Problem:** We've shown that the kernel of any action by $G$ on a set $X$ is a normal subgroup of $G$. (a)-(d) are easy, once you understand them.

**(a)** Let $H < G$ and let $G$ act on the left cosets of $H$ by left multiplication. Show that $K$, the kernel of the action, is a normal subgroup of $H$.

**(b)** Suppose that $H$ has index $n$ and that $G$ is simple. Use part (a) to show that $G$ can be embedded in $S_n$.

**(c)** Continuing (b), show that $|G:K|$ divides $n!$.

**(d)** Let $H < G$ have index $n$. If $|G|$ does not divide $n!$ then $G$ is not simple.

**(e)** Use the existence of a Sylow subgroup to show that there are no simple groups of order 12, 20, 24, 28, 36,..., and none of order $p^a q$ for distinct primes $p$ and $q$ such that $p^a < q$.

**Problem 6:** Suppose $G$ is abelian and $f : G \to \mathbb{Z}$ is surjective. Let $K$ be the kernel. Show $G$ has a subgroup $H$ isomorphic to $\mathbb{Z}$ and $G \cong H \oplus K$.

**Problem 7:** Consider the following subgroups of $\mathrm{Gl}(F, n)$:

$$\mathrm{Sl}(F,n) \qquad \mathrm{Diag}(F,n) \qquad F^* I_n$$

(a) Show that $\mathrm{Sl}(F,n)$ and $F^* I_n$ are normal in $\mathrm{Gl}(F,n)$ but that $\mathrm{Diag}(F,n)$ is not, except in one very special case.
(b) Is it true that $\mathrm{Gl}(F,n)$ is the product of $\mathrm{Sl}(F,n)$ and $F^* I_n$? The answer is subtle—it depends on the field and on $n$!
To get started you might want to try some small fields, like $\mathbb{F}_3$, using a computer algebra system.