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Math 627B: Modern Algebra II
Homework VI

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Most of these problems have short solutions. Some are straightforward, some require a bit of cleverness. 3(d) is probably the hardest.

Your solutions should be understandable by a peer, so, not every detail has to be explained, provided a peer would know how to fill in the details. This is the art of exposition, knowing your audience and how to succintly communicate essentials.

Feel free to ask questions over break via email.

Problem 3: A group is *metabelian* when it has a normal subgroup N such that N and G/N are both abelian. A group is *metacyclic* when it has a normal subgroup N such that N and G/N are both cyclic.

(a) Show that S_3 is metacyclic.

(b) Show that A_4 is metabelian but not metacyclic (you may refer to the lattice of subgroups that you derived previously).

(c) Prove that any subgroup of a metabelian group is also metabelian.

(d) Prove that any quotient group of a metabelian group is metabelian. [Look carefully at the proof of the 2nd isomorphism theorem and adapt it to this question.]

Problem 4: Let F be a field and let G be the subgroup of $\mathrm{Gl}(F,n)$ that stabilizes the standard basis vector $\begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$

(a) Show that G has a subgroup H isomorphic to $\mathrm{Gl}(F, n-1)$.

(b) Show that G has a normal subgroup N isomorphic to the additive group F^{n-1}.

(c) Show G is the semidirect product $N \rtimes H$.

Problem 5: Some normal subgroups.

(a) Let H be a subgroup of G and $N = \bigcap_{a\in G} a^{-1}Ha$.
Prove that N is normal in G.

(b) Let H be the intersection of all subgroups of G of order n.
Prove that H is normal in G.

Problem 6: Let K be a subgroup of G and define

$$N_G(K) = \{g \in G : gKg^{-1} = K\}$$
$$C_G(K) = \{g \in G : gkg^{-1} = k \text{ for all } k \in K\}$$

(a) Show that K is a normal subgroup of $N_G(K)$.

(b) If $H \leq G$ and K is a normal subgroup of H show that $H < N_G(H)$. So, $N_G(K)$ is the largest subgroup of G in which K is normal.

(c) Show that $C_G(K)$ is a normal subgroup of $N_G(K)$.

(d) Show that $N_G(K)/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$.

Problem 7: Let A be an abelian group and let $|A| = mn$ with m and n coprime. Let

$$mA = \{ma : a \in A\} \qquad \text{and} \qquad A[m] = \{a \in A : ma = 0\}$$

(a) Prove that $mA = A[n]$ and that this is a subgroup of A.

(b) Prove that $A \cong mA \times nA$. (See Ash p. 20 top.)