Math 621–Advanced Topics in Algebra: Galois Theory and Related Topics

Spring 2024 Syllabus

Schedule: Tuesday, Thursday, 12:30 -1:45, January 18 to May 2 in GMCS 405.
I will be honoring the strike.
If there are outbreaks of illness a few days may be online.
Holidays: Spring break April 1-5.
Final: Thursday, May 9, 10:30-12:30.

Instructor: Michael E. O'Sullivan (he).
You may call me Mike or something more formal if you prefer.
Office Hours: GMCS 582 Tu, Th 1:45-3:00.
Other times TBD; also by appointment.
I enjoy meeting with students, so don't hesitate to ask.
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Course Description: The main goal of this course is to introduce the main theorems and standard examples in Galois theory. Along the way we will do some more advanced group theory (group actions, Sylow theorems, composition series, group extensions). We will focus on interesting examples of the Galois correspondence for finite fields, number fields and function fields.

The roots of Galois theory lead back to problems posed by the ancient Greeks and their predecessors. Greek geometers achieved remarkable constructions with ruler and compass, but a number of simple, nagging, problems remained unresolved until the Renaissance. For example: Is it possible to trisect an arbitrary angle? Which regular polygons are constructible? In algebra, several civilizations investigated the solution of a quadratic equation. The attempt to find a solution for higher degree equations was another project that occupied numerous mathematicians. The resolution of these ancient questions culminated in Galois' theory of fields. It is a delightful subject, and the modern treatment highlights the interplay between three key areas of algebra: groups, rings and fields.

A second goal of this course is to develop some of the main results for multivariate polynomial rings over a field. Fundamental theorems for univariate polynomial rings over a field ($k[x]$) parallel those for the integers: the quotient remainder theorem, the greatest common divisor theorem, the Euclidean algorithm, unique factorization. Interestingly, $k[x]$ has additional properties that make some problems more tractable over $k[x]$ than they are over $\mathbb{Z}$. We will explore some extensions of the fundamental theorems, with appropriate modification, to multivariate polynomial rings over a field.

Prerequisites: A good understanding of the basics of groups, rings and fields (Math 620 is sufficient). Particularly this material:

- Properties of the Integers: The division theorem and divisibility, greatest common divisor, the Euclidean algorithm, unique factorization, modular arithmetc.
- Polynomial Ring in One Variable: The division theorem, greatest common divisor, the Euclidean algorithm, unique factorization. The correspondence between factors and roots. Polynomial rings modulo a polynomial.
- Commutative Rings and ideals: The general language of rings and ideals. Integral domains, the quotient of an integral domain by an ideal, homomorphisms.
- Linear Algebra: nullspace, subspace, dimension, basis.

Course Materials. I am writing lecture notes for the course, and will distribute them as I finalize them. You might want a physical copy of the Hungerford book (a cheap, 2nd edition) as a reference with more detail than my notes. Dummit & Foote's book is a great resource if you plan to continue studies in algebra.

- Thomas Hungerford: *Abstract Algebra: An Introduction.* This is the textbook we use for our undergraduate algebra courses. It is well written and should be a nice reference to flesh out details and give examples.
- David Dummit, Richard Foote, *Abstract Algebra.* A massive standard reference for graduate level algebra, full of examples and detailed proofs.
- Robert Ash: *Abstract Algebra: The Basic Graduate Year.* Available for free at `https://web.archive.org/web/20230326022245/https://faculty.math.illinois.edu/~r-ash/Algebra.html` Physical copies are published by Dover and are cheap.
- William A. Stein et al. Sage Mathematics Software The Sage Development Team, 2011, `https:www.sagemath.org`
- SDSU SageMath Tutorials `https://mosullivan.sdsu.edu/Teaching/sdsu-sage-tutorial/index.html` `https://doc.sagemath.org/html/en/thematic_tutorials/index.html`
- Magma online calculator. `http://magma.maths.usyd.edu.au/calc/`

Learning Outcomes:. It is standard these days to have learning outcomes for every course; rather than simply listing the topics covered. My approach to this is as follows. In every math course that I teach, I want students to advance in the skills listed below (adapted from the Degree Learning Outcomes for the SDSU math major as presented on the department website). In this course we do this work in the theory of field extensions and their Galois groups, and properties of multivariate polynomial rings over a field.

Foundational knowledge. State major definitions, axioms, and theorems and use examples to illustrate.

Use logical reasoning. Read a proof and explain the logic and derivations. Write a mathematical proof using an appropriate method.

Use algebraic tools and methods. Derive answers, apply algorithms, and compute, both by hand and using mathematics software.

Explore mathematical ideas independently. Have confidence to read challenging material that is beyond that explored in a textbook or class.

Communicate mathematical ideas effectively. Make progress toward the mathematicians goal: writing that gets to the essence of the matter and is brief, clear, and polished.

Nurture the learning of others. Work with others in a way that is collegial, inclusive and empowering. Contribute, but seek understanding of other perspectives.

Format: Class time will mix lecture with problem solving. We may also spend some time using SageMath in the computer lab. I will assign specific pages from my notes to be read before class. You may not understand some of the material, but, read the assigned pages, formulate questions that you have, and be prepared to discuss this in class. I have some short ($5 - 15$ min.) recorded lectures on some topics, which will free class time for discussion of problems. Be prepared to present your work in class and also to work on problems in class.

Communications: Please see my website for detailed information about the course. My "Teaching" page has a link to the course page, which has lots of information and links to assignments and to a schedule of topics.

I will set up a Discord server for the class. There is an initial invitation that just gets you access to the class. When you first join, you will not be able to post. Send me a Discord message @mosullivan-math with your full name (first & last) and RedID number, and I will change your role to @students, giving you posting permission.

Please feel free to post questions and to answer questions. Relative to HW problems, provide guidance not solutions. As with all interactions in the course: Nurture the learning of others. Work with others in a way that is collegial, inclusive and empowering. Contribute, but seek understanding of other perspectives.

I will sometimes use Canvas to communicate to the whole class, to post grades, and for miscellaneous other things. To contact me it is best to use the email address above.

It is extremely important to me that all students feel welcome and supported in the class. Please let me know about the name and pronouns you use. Feel free to communicate with me about any concerns you have with the class environment.

Grading: We will have weekly assignments, two midterms and a final project. For the weekly assignments, there will be a small number of problems (10 or so) which you should write up carefully in LaTeX. You may draw diagrams or write other things that have difficult layout by hand, neatly.

Point value for the work will be as follows (plus or minus 50 points for any one item.)

Weekly work	400
Tests (2)	300
Final Project	300
Total	1000

Grading Scale: A: 100-85%, B: 84-75%, C: 74-65%, D: 65-50%, F: below 50%.

Collaboration Policy: You are encouraged to work together to solve problems, but you should write the solutions individually. On homework and tests, your solutions should be understandable by a peer. Not every detail has to be explained, provided a peer would know how to fill in the details. This is the art of exposition, knowing your audience and how to succinctly communicate essentials.