Problem Set 1

Problems with (HW) are due Thursday 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well.

Exercises 1.1. Basic Group Properties. Let $G, *$ be a group. Give succinct but complete proofs of the following.

- **(a)** The identity element is unique.
- **(b)** The inverse of any element is unique.
- **(c)** The cancellation law holds: $a*b = a*c$ implies $b = c$ (and similarly for cancellation on the right).
- **(d)** If $a * g = g$ for some $g \in G$, then $a = e_G$.
- **(e)** $(a * b)^{-1} = b^{-1} * a^{-1}$.
- **(f)** $(a^{-1})^{-1} = a$.
- **(g)** If every element is of order 2, then G is abelian.

Exercises 1.2. A Weird Group

- **(a)** Show that $\mathbb{Z}$ is a group under the operation $*$ (not the usual multiplication) defined by $a * b = a + b - 2$. (What is the identity element? What is the inverse of an element a?)
- **(b)** (HW) Find an isomorphism from $\mathbb{Z}, +$ to $\mathbb{Z}, *$.

Exercises 1.3. The Dihedral Group D_5. This is 2.1.9 in the notes.

- **(a)** Find a formula for rt_i.
- **(b)** (HW) Find a formula for $r^j t_i$.

Exercises 1.4. Lattice Diagrams for Groups

- **(a)** Draw the subgroup lattice diagram for $\mathbb{Z}_{45}$.
- **(b)** Draw the subgroup lattice diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- **(c)** Draw the subgroup lattice diagram for $\mathbb{Z}_2 \times \mathbb{Z}_3$.
- **(d)** (HW) Draw the subgroup lattice diagram for $\mathbb{Z}_2 \times \mathbb{Z}_4$.
- **(e)** (HW) Draw the subgroup lattice diagram for $\mathbb{Z}_3 \times \mathbb{Z}_4$.
- **(f)** (HW) Find all subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$. Describe the logic of your process for finding them. Present them in an organized fashion. Draw the lattice if you can.

Exercises 1.5. (HW) This is 2.2.11 in the notes

- **(a)** If $g \in G$ has order m and $h \in H$ has order n, find the order of $(g, h) \in G \times H$.
- **(b)** Suppose that $a, b \in G$ commute (that is $ab = ba$). If $\operatorname{ord}(a)$ and $\operatorname{ord}(b)$ are coprime find the order of ab.
- **(c)** Let A be an abelian group with finite exponent. Show that there is some $a \in A$ such that $\operatorname{ord}(a) = \exp(A)$.