Problem Set 2

Problems with (HW) are due Tuesday 9/13 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 2.1. Automorphism groups

- **(a)** Show that $\mathrm{Aut}(\mathbb{Z})$ has two elements and $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$.
- **(b)** Compute $\mathrm{Aut}(\mathbb{Z}_n)$ for $n = 2, 3, 4, 5, 6, 7$. [For each the answer is a cyclic group.]
- **(c)** (HW) Show that $\mathrm{Aut}(\mathbb{Z}_8)$ is not cyclic.

Exercises 2.2. Lattice Diagrams for Groups

- **(a)** Identify the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ that are of the form $H \times K$ with H a subgroup of $\mathbb{Z}_2$ and K a subgroup of $\mathbb{Z}_4$.
- **(b)** Identify the subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ that are of the form $H \times K$ with H a subgroup of $\mathbb{Z}_4$ and K a subgroup of $\mathbb{Z}_4$.
- **(c)** Draw the subgroup lattice for S_3.
- **(d)** (HW) Draw the subgroup lattice diagram for D_4.
- **(e)** (HW) Draw the subgroup lattice diagram for D_5.

Exercises 2.3. (HW)Some subgroups of abelian groups. Let A be an abelian group and let m be an integer.

- **(a)** Let $mA = \{ma : a \in A\}$. Show that mA is a subgroup of A.
- **(b)** Let $A[m] = \{a \in A : ma = 0\}$. Show that $A[m]$ is a subgroup of A.
- **(c)** Give an example in which $mA \cap A[m]$ is trivial (just 0) and given an example in which it is not trivial.

Exercises 2.4. (HW) Let H be a subgroup of a group G.

- **(a)** For any $a \in G$ show that $\left\{aha^{-1} : h \in H\right\}$ is a subgroup of G. This is often written aHa^{-1}.
- **(b)** Show that there is an isomorphism between H and aHa^{-1}.

Exercises 2.5.

- **(a)** For $\pi \in S_n$, the sum of the signature list is n.
- **(b)** If $\pi = \sigma_1 \sigma_2 \cdots \sigma_r$ is a cycle decomposition, then $\pi^k = \sigma_1^k \sigma_2^k \cdots \sigma_r^k$. Under what conditions is this also a cycle decomposition in the sense that each Σ_i^k is a cycle?
- **(c)** The order of $\pi \in S_n$ is the lcm of the signature list.
- **(d)** Identify all possible signatures for elements of S_4 and the order of these elements. What is the exponent of S_4?
- **(e)** (HW) Identify all possible signatures for elements of S_5 and the order of these elements. What is the exponent of S_5?
- **(f)** (HW) For each possible signature in S_5, count how many elements have that signature. Check that you get the correct total number of elements in S_5.