Problem Set 12

Here are some problems that we will do this week. Something like one of these may appear on the final test.

Problems 12.1. (Discussion) Minimal polynomials and field extensions.

(1) Find the minimal polynomials of $3\sqrt{2}+4$, and $\sqrt[3]{2}+1$ over $\mathbb{Q}$. (Look for a quick and clever solution.)

(2) Find the minimal polynomial of $\alpha = \sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$. (Find a linear combination of $\alpha^4, \alpha^2, \alpha^2, \alpha, 1$ that adds to 0.)

(3) Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. (Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$ and get a contradiction. You know that $\sqrt{p}$ is irrational for a prime p.)

(4) Argue that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(5) (HW) Find all 3 proper subfields of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ (aside from $\mathbb{Q}$).

Problems 12.2. (HW) Splitting fields.

(1) Find the splitting field of $(x^2-5)(x^2-7)$ and all its subfields (3 proper subfields).

(2) Find the splitting field of x^4-4 over $\mathbb{Q}$ and all its proper subfields (there are 3 other than $\mathbb{Q}$).

(3) Find the splitting field of x^d-1 over $\mathbb{Q}$ for $d=2$, 3, 4, 5 (harder!), and 6.

Problems 12.3. (HW) We studied the splitting field of x^4+1 over $\mathbb{Q}$ showing that it is $\mathbb{Q}(\alpha)$ where $\alpha = \frac{\sqrt{2}}{2}(1+i)$.

(1) Factor x^4+1 in $\mathbb{Q}(\alpha)$.

(2) There are 3 different ways to write x^4+1 as a product of quadratics (using complex coefficients). Find them.

(3) There are 3 different subfields of $\mathbb{Q}(\alpha)$. Find them.

(4) Explain the relationship between the factorizations of x^4+1 that you found in (b) and the fields you found in (c).

Problems 12.4. We found the splitting field of x^3-2 over $\mathbb{Q}$ in class.

(1) This field has four proper subfields besides $\mathbb{Q}$ itself. Find them and find their dimensions over $\mathbb{Q}$.

(2) (HW) Factor x^3-2 over each of these subfields.