Problem Set 2

Problems with (HW) are due Friday 9/20 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Problems 2.1. Lattice Diagrams for Groups

(1) Draw the lattice diagram for $\mathbb{Z}_2 \times \mathbb{Z}_4$. Identify the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ that are of the form $H \times K$ with H a subgroup of $\mathbb{Z}_2$ and K a subgroup of $\mathbb{Z}_4$.

(2) Draw the subgroup lattice diagram for D_5.

(3) (HW) Identify the subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ and draw its lattice diagram. Identify the subgroups that are of the form $H \times K$ with H a subgroup of $\mathbb{Z}_4$ and K a subgroup of $\mathbb{Z}_4$.

(4) (HW) Draw the subgroup lattice diagram for D_4.

Problems 2.2. Automorphism groups

(1) Show that $\mathrm{Aut}(\mathbb{Z})$ has two elements and $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$.

(2) Compute $\mathrm{Aut}(\mathbb{Z}_n)$ for $n = 2, 3, 4, 5, 6, 7$. [For each the answer is a cyclic group.]

(3) (HW) Show that $\mathrm{Aut}(\mathbb{Z}_8)$ is not cyclic.

Problems 2.3. (HW) Let H be a subgroup of a group G.

(1) For any $a \in G$ show that $\{aha^{-1} : h \in H\}$ is a subgroup of G. This is often written aHa^{-1}.

(2) Show that the following function is an automorphism of G.

$$\begin{aligned} \varphi_a : G &\longrightarrow G \\ g &\longmapsto aga^{-1} \end{aligned}$$

(3) Show that there is an isomorphism between H and aHa^{-1}.

Problems 2.4. Signatures

(1) For $\pi \in S_n$, the sum of the signature list is n.

(2) If $\pi = \sigma_1 \sigma_2 \cdots \sigma_r$ is a cycle decomposition, then $\pi^k = \sigma_1^k \sigma_2^k \cdots \sigma_r^k$. Under what conditions is this also a cycle decomposition in the sense that each Σ_i^k is a cycle?

(3) The order of $\pi \in S_n$ is the lcm of the signature list.

(4) Identify all possible signatures for elements of S_4 and the order of these elements. What is the exponent of S_4?

(5) (HW) Identify all possible signatures for elements of S_5 and the order of these elements. What is the exponent of S_5?

(6) (HW) For each possible signature in S_5, count how many elements have that signature. Check that you get the correct total number of elements in S_5.

Problems 2.5. (HW) Generators for S_n.

(1) Show that S_n is generated by the $n-1$ elements $(1,k)$ for $k=2,\dots,n$. [Show that you can get an arbitrary transposition by conjugating $(1,k)$ by some $(1,j)$.]

(2) Show that S_n is generated by 2 elements: $(1,2)$ and $(1,2,3,\dots,n-1,n)$. [Show that you can get all $(1,k)$ from these two using conjugation.]

Problems 2.6. Generators for A_n.

(1) Suppose that σ is a k-cycle and τ is an m-cycle and there is exactly one element of $\{1,\dots,n\}$ that is in the support of both σ and τ. Show that $\sigma\tau$ is a $(k+m-1)$-cycle.

(2) Show that the product of two disjoint transpositions can also be written as the product of two 3-cycles.

(3) (HW) Use part (a) (with $k=m=2$) and part (b) to prove that A_n is generated by 3 cycles.

(4) (HW) Compute $(1,2,a)(1,b,2)$ for a,b distinct and not equal to 1 or 2. . Use the result as motivation to show that the 3-cycles of the form $(1,2,a)$ generate A_n for $n \geq 4$.