# Problem Set 3

exercises with (HW) are due Friday 9/27. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

*Exercises* 3.1. Generators for $A_n$.

- **(a)** Suppose that $\sigma$ is a $k$-cycle and $\tau$ is an $m$-cycle and there is exactly one element of $\{1,\dots,n\}$ that is in the support of both $\sigma$ and $\tau$. Show that $\sigma\tau$ is a $(k+m-1)$-cycle.
- **(b)** Show that the product of two disjoint transpositions can also be written as the product of two 3-cycles.
- **(c)** (HW) Use part (a) (with $k=m=2$) and part (b) to prove that $A_n$ is generated by 3 cycles.
- **(d)** (HW) Compute $(1,2,a)(1,b,2)$ for $a,b$ distinct and not equal to 1 or 2. Use the result as motivation to show that the 3-cycles of the form $(1,2,a)$ generate $A_n$ for $n \geq 4$.

*Exercises* 3.2. Cayley's Theorem

- **(a)** Let $n=5$ and think of $\mathbb{Z}_n$ in the usual way as $\{0,1,2,3,4\}$ with addition modulo $n$. For each $a \in \mathbb{Z}_n$ write down in tabular form the function on $\mathbb{Z}_n$ defined by addition of $a$.
- **(b)** Show that part (a) defines a function from $\mathbb{Z}_5$ to $S_5$, provided you think of $S_5$ as the group of permutations of $\{0,1,2,3,4\}$. Show that this function is a homomorphism.
- **(c)** Now consider $\mathbb{Z}_2 \times \mathbb{Z}_2$. Enumerate the 4 elements in any way you choose as $a_1,a_2,a_3,a_4$. For each $a_i$ define a permutation $\sigma_i$ by $a_ia_1 = a_{\sigma_i(1)}$, $a_ia_2 = a_{\sigma_i(2)}$, $a_ia_3 = a_{\sigma_i(3)}$, $a_ia_4 = a_{\sigma_i(4)}$.
- **(d)** Show in part (c) that this gives a homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $S_4$.
- **(e)** (HW) Similar to the above examples, the next steps define a homomorphism from $D_3$ to $S_6$. Enumerate the elements of as follows $D_3 = \left\{a_1 = r^0, a_2, = r, a_3 = r^2, a_4 = t, a_5 = rt, a_6 = r^2t\right\}$ For each $a_i$ define a permutation $\sigma_i$ in $S_6$. $\sigma_1$ is the identity, and $\sigma_2$ is given by $\sigma_2(i) = k$ whenever $ra_i = a_k$. Verify that each $\sigma_i$ is indeed a permutation by writing it in permutation notation.
- **(f)** (HW) Verify in three examples: for any $a,b \in D_3$, the permutation corresponding to $ab$ equals the product of the permutations corresponding to $a$ and $b$.
- **(g)** (HW) Which elements of $D_3$ correspond to odd permutations in $S_6$?

*Exercises* 3.3. More examples of conjugation.

- **(a)** Show that $A_n$ is invariant under conjugation: for any $\pi \in S_n$, $\pi A_n \pi^{-1} = A_n$.
- **(b)** (HW) Let $C_n$ be the rotation subgroup of $D_n$. Find two elements of $C_4$ that are conjugate as elements of $D_4$ but are not conjugate as elements of $C_4$.
- **(c)** (HW) Find two elements of $D_4$ (the subgroup generated by $(1,2,3,4)$ and $(1,2)(3,4)$) that are conjugate as elements of $S_4$ but are not conjugate as elements of $D_4$. A computer algebra system will be useful.

*Exercises* 3.4. (HW) Inner Automorphisms

For $a$ an element of a group $G$, define a function $\varphi_a : G \longrightarrow G$ by $\varphi_a(g) = aga^{-1}$.

**(a)** We have shown that $\varphi_a$ is an automorphism of $G$. Show that $\varphi : G \longrightarrow \mathrm{Aut}(G)$ defined by $\varphi : a \longmapsto \varphi_a$ is a homomorphism. The image, $\{\varphi_a : a \in G\}$, is therefore a subgroup of $\mathrm{Aut}(G)$. It is called $\mathrm{Inn}(G)$, the group of **inner automorphisms** of $G$.

**(b)** What is the kernel of $\varphi$?

*Exercises* 3.5. Challenge problem

**(a)** Let $n$ be a positive integer and $k > n/2$. Find a formula for the number of elements of $S_n$ that include a $k$-cycle.

**(b)** Use Stirling's formula to approximate the formula you just computed.

**(c)** Estimate the probability that a random element of $S_n$ has a cycle of length larger than $n/2$.