Problem Set 4

Problems with (HW) are due Friday 10/4. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Problems 4.1. (HW) Recall that the exponent of a group G is the lcm of the orders of the elements (if this is finite).

(1) For a finite group G show that the the exponent of G divides the order of G. [This takes some careful application of Lagrange's theorem.]

(2) Give an example to show that there may not be an element in G whose order is the exponent of G.

Problems 4.2. (HW) Conjugation and cycle decomposition.

Consider conjugation by $\pi \in S_n$.

(1) Let $(a_1, a_2, \ldots, a_k) \in S_n$ be a k-cycle, so the a_i are distinct. Show that

$$\pi * (a_1, a_2, \ldots, a_k) * \pi^{-1} = \Big(\pi(a_1), \pi(a_2), \ldots, \pi(a_k)\Big)$$

[Consider two cases, $b = \pi(a_i)$ for some i, and $b \notin \{\pi(a_1), \pi(a_2), \ldots \pi(a_k)\}$. Explain why this breakdown into two cases makes sense.]

(2) If A and B are disjoint subsets of $\{1, \ldots, n\}$ show that $\pi(A)$ and $\pi(B)$ are also disjoint.

(3) If $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ is the cycle decomposition of σ, find the cycle decomposition of $\pi\sigma\pi^{-1}$ and justify your answer.

(4) Conclude that the conjugation of any $\sigma \in S_n$ by π has the same signature as σ.

Problems 4.3. Key examples of normal subgroups.

(1) Show that any subgroup of index 2 is normal.

(2) Prove that the center of a group G is a normal subgroup of G.

(3) Let N be a normal subgroup of G. For any subgroup H of G, $H \cap N$ is a normal subgroup of H.

(4) Show that the intersection of two normal subgroups of G is normal in G. (Take a step further to show that the intersection of any collection of normal subgroups of G is again a normal subgroup of G.)

(5) If $\varphi : G \longrightarrow H$ is a homomorphism and N is normal in H, then $\varphi^{-1}(N)$ is normal in G.

(6) If $\varphi : G \longrightarrow H$ is a *surjective* homomorphism and M is normal in G, then $\varphi(M)$ is normal in H.

Problems 4.4. (HW) Even more normal subgroups

(1) Let G be a group, possibly infinite. Let I be some indexing set and for each $i \in I$ let H_i be a subgroup of G. Prove that for any $a \in G$,

$$a\Big(\bigcap_{i\in I} H_i\Big)a^{-1} = \bigcap_{i\in I} aH_ia^{-1}$$

(2) Let H be a subgroup of G and let $N = \bigcap_{g\in G} g^{-1}Hg$. Prove that N is normal in G.

(3) Let $n \in \mathbb{N}$ and let K be the intersection of all subgroups of G of order n. Prove that K is normal in G.

Problems 4.5. The **quaternion group** is defined by

$$Q = \langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^{-1}b\rangle$$

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(1) Show that Q has 8 elements. List them in a useful fashion and show how to multiply them as we did for the dihedral group.

(2) Show that Q has 1 element of order 2 and 6 of order 4.

(3) Draw the lattice diagram for this group.

(4) Identify the normal subgroups of Q.

Problems 4.6. Recall the lattice for $\mathbb{Z}_2\times\mathbb{Z}_4$. (Or $\mathbb{Z}_4\times\mathbb{Z}_4$ for a bigger challenge.)

(1) For each subgroup, H of $\mathbb{Z}_2 \times \mathbb{Z}_4$, find the lattice for the quotient group $\mathbb{Z}_2 \times \mathbb{Z}_4/H$. What familiar group is the quotient group isomorphic to?

(2) For each subgroup H, find generators for $\mathbb{Z}_2 \times \mathbb{Z}_4/H$.