Problem Set 5

Problems with (HW) are due Friday 10/18. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Let G, H be groups.

- The **external direct product** of G and H is $G \times H$ with component-wise multiplication.
- G is the **internal direct product** of two subgroups N and K when both are normal in G and
$$G = NK \qquad \text{and} \qquad N \cap K = \{e\}$$
In this case Corollary 3.1.6 says $N \times K \cong G$ (using the function $(n, k) \longmapsto nk$). Example: for k, n coprime, $\mathbb{Z}_{kn}$ is the internal direct product of $\langle k \rangle$ (which is isomorphic to $\mathbb{Z}_n$) and $\langle n \rangle$, which is isomorphic to $\mathbb{Z}_k$.
- G is the **internal semidirect product** of two subgroups N and H when N is normal in G (and is not necessarily) and
$$G = NH \qquad \text{and} \qquad N \cap H = \{e\}$$
- To define an **external semidirect product** of two arbitrary groups N and H it is necessary to specify a homomorphism $\varphi : H \longrightarrow \text{Aut}(N)$. The external semidirect product defined by φ is $N \times H$ with a twisted multiplication, $*$.
$$(n_1, h_1) * (n_2, h_2) = (n_1 \varphi_{h_1}(n_2),\, h_1 h_2)$$
Here φ_{h_1} is the image of h_i under the homomorphism φ, so it is an automorphism of N. We write $N \rtimes_\varphi H$.

The first suite of exercises concern some familiar groups that are internal semidirect products. You are invited (encouraged, coerced ;-) into thinking of them as external semi-direct products. Basically, every answer is the same, use the word conjugation. Do some example computations.

Exercises 5.1. Some familiar semidirect products.

Several familiar groups are internal semidirect products of much simpler groups. Verify that each of the following is an internal semidirect product of the two given subgroups by using Corollary 3.1.4. In each case identify the homorphism from one subgroup to the automorphism group of the other.

(a) D_n is the semidirect product of its rotation group, $\langle r \rangle$, and any subgroup generated by a reflection, t. There is an implicit homomorphism $\varphi : \langle t \rangle \longrightarrow \text{Aut}(\langle r \rangle)$. What is it?

(b) $S_n = A_n \rtimes \langle (1, 2) \rangle$.

(c) $S_4 = V \rtimes S_3$ where V is Klein-4 subgroup with elements of the form $(a, b)(c, d)$ with a, b, c, d distinct elements of $\{1, 2, 3, 4\}$.

(d) In $\mathrm{GL}_n(F)$, for F a field, let T be the upper triangular matrices with nonzeros on the diagonal; let U be the upper triangular matrices with 1's on the diagonal and let D be the diagonal matrices with nonzero elements on the diagonal.
For $n = 2$, show that $T = U \rtimes D$.

(e) Do the previous problem for arbitrary n.

Exercises 5.2. The definition of semidirect product makes sense!
Using the notation for the external semidirect product, verify the following.

(a) (e_H, e_N) is the identity element.

(b) Each element does have an inverse.

(c) (HW) The associative law holds. [Explain carefully each step.]

(d) (HW) $N \times \{e_H\}$ is a normal subgroup of $N \rtimes H$.

Exercises 5.3. (HW) Semidirect products and matrix groups
Let F be a field. Let $\mathrm{GL}_n(F)$ be the general linear group: $n \times n$ matrices over F with nonzero determinant. Let $\mathrm{SL}_n(F))$ be the special linear group: $n \times n$ matrices with determinant 1. Let F^*I be the nonzero multiples of the identity matrix. In this problem we investigate the finite fields F and values of n for which $\mathrm{GL}_n(F) \cong \mathrm{SL}_n(F) \times F^*I$.

(a) For the fields $F = \mathbb{F}_3$ and $F = \mathbb{F}_5$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n odd, but not for n even.

(b) For the field $F = \mathbb{F}_7$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n coprime to 6, and is not otherwise.

(c) (Challenge) For which prime numbers p and which n is $\mathrm{GL}_n(\mathbb{F}_p)$ a direct product as above?

Exercises 5.4. (HW) External semidirect products of cyclic groups.

(a) Use the definition of external semidirect product to create the other nonabelian group of order 12 (besides D_6 and A_4), $C_3 \rtimes_\varphi C_4$ where φ is the only possible map $C_4 \longrightarrow \mathrm{Aut}(C_3)$ that is not trivial. Let a be the generator for C_3 and b the generator for C_4. Show the following:

(1) Every element can be represented uniquely as (a^i, b^j) for $i \in \{0, 1, 2\}$ and $b \in \{0, 1, 2, 3\}$

(2) Find a general formula for $(a^i, b^j)(a^m, b^n)$. It may be useful to break this into cases.

(3) Find the inverse of (a^i, b^j).

(4) Show that the group you just created is isomorphic to the group presented as $\langle a, b | a^3 = b^4 = 1, ba = a^2 b\rangle$

(b) Use the definition of external semi-direct product to create the only nonabelian group of order 21 (the smallest non-abelian group of odd order), $C_7 \rtimes C_3$. Let a be the generator for C_7 and b the generator for C_3. Show how to represent, multiply, and invert elements of this group as you did in part (a).

(c) (Challenge Problem) Use the definition of external semi-direct product to construct other semi-direct products $C_m \rtimes C_n$. You will need to start with a homomorphism $\varphi : C_n \longrightarrow \mathrm{Aut}(C_m)$. See how many of the small non-abelian groups you can find in the table of small abelian groups on Wikipedia.