Problem Set 6

Probtlems with (HW) are due Friday 10/18. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 6.1. The Classification of Finite Abelian Groups.

- **(a)** Find the elementary divisors and the invariant factors for $\mathbb{Z}/30 \times \mathbb{Z}/600 \times \mathbb{Z}/420$.
- **(b)** Find the elementary divisors and the invariant factors for $\mathbb{Z}/50 \times \mathbb{Z}/75 \times \mathbb{Z}/136 \times \mathbb{Z}/21000$.
- **(c)** (†) Let $n = p^6 q^5 r^4$ where p, q, r are distinct primes. How many abelian groups are there of order n?
- **(d)** How many have exactly two invariant factors?
- **(e)** (HW) For n as in (c), how many abelian groups of order n have k invariant factors, for $k = 1, 2, 3, 4, 5, 6$? Check that that the total of these values is the same as the response to (c).

There is a more general theorem than we treated in class. A group A is **finitely generated** if $A = \langle a_1, \ldots, a_s \rangle$ for some finite set of elements $\{a_1, \ldots, a_s\}$ in A.

Theorem 6.2. *Any finitely generated abelian group is isomorphic to the direct product of cyclic groups, each of which is either $\mathbb{Z}$ or of prime power order. This decomposition is unique, up to reordering the factors.*

The following two exercises give some pieces of the proof.

Exercises 6.3. The torsion subgroup of an abelian group.

Let A be an infinite abelian group. Let $\mathrm{Tor}(A)$ be the set of elements with finite order, which is called the **torsion subgroup** of A.

- **(a)** (†) Show that $\mathrm{Tor}(A)$ is, indeed, a normal subgroup of A.
- **(b)** (†) Show that $\mathrm{Tor}(A) = \bigcup_{m \in \mathbb{N}} A[m]$. (Note that, even inside an abelian group, the union of subgroups is not usually a group!)
- **(c)** (HW) Show that $\mathrm{Tor}(A/\mathrm{Tor}(A))$ is trivial. That is, letting $T = \mathrm{Tor}(A)$, the only element of finite order in A/T is the identity element, $0 + T$.
- **(d)** (HW) Give an example (a simple one!) of a finitely generated abelian group in which the identity element together with the elements of infinite order do *not* form a subgroup. (As opposed to the torsion subgroup.)

We say that A is **torsion free** when its torsion subgroup is trivial. The proof above shows that, for A an abelian group, $A/\mathrm{Tor}(A)$ is torsion free.

The next step is to show that a finitely generated, torsion free abelian group is isomorphic to $\mathbb{Z}^r$. That implies that $A/\mathrm{Tor}(A) \cong \mathbb{Z}^r$.

The final step is to show that A has a subgroup isomorphic to $\mathbb{Z}^r$. Here is a key piece.

Exercises 6.4. (HW) Toward classifying finitely generated abelian groups.

(a) (†) Let A be an abelian group. Suppose $f : A \to \mathbb{Z}$ is a surjective homomorphism with kernel K. Show that A has an element a such that A is the internal direct product $K \times \langle a \rangle$.

(b) In the previous problem, suppose f is not surjective but $f(A) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Show that it still holds that there is an element $a \in A$ such that A is the internal direct product $K \times \langle a \rangle$.

Exercises 6.5. Infinitely generated abelian groups, much more complicated. Consider the group $\mathbb{Q}/\mathbb{Z}, +$.

(a) On a number line, sketch a region that contains exactly one element for each equivalence class of $\mathbb{Q}/\mathbb{Z}$.

(b) Show that for any integer n there is an element of order n in $\mathbb{Q}/\mathbb{Z}$.

(c) (HW) How many elements of order n are there in $\mathbb{Q}/\mathbb{Z}$?

(d) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

(e) (HW) Show that every nontrivial cyclic subgroup is generated by $\frac{1}{n}$ for some integer $n > 1$.

(f) (HW) Show that $\mathbb{Q}/\mathbb{Z}$ is not finitely generated as an abelian group.

(g) (Challenge) Show that $\mathbb{Q}/\mathbb{Z}$ cannot be written as a direct product of $\langle a \rangle$ and another group H for any nonzero $a \in \mathbb{Q}/\mathbb{Z}$.