# Problem Set 9

Problems with (HW) are due 11/25 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

*Problems* 9.1. Consider the following

- $R = \left\{a/2^i : a \in \mathbb{Z}, i \in \mathbb{N}_0\right\}$.
- $S = \{a/b : a \in \mathbb{Z} \text{ and } b \text{ is an odd integer}\}$.
- $T = \left\{a/100^i : a \in \mathbb{Z}, i \in \mathbb{N}_0\right\}$. (HW)

**(1)** Show that each of these is a subring of $\mathbb{Q}$ containing $\mathbb{Z}$.
**(2)** Identify all the units in each of these rings.
**(3)** Show that in each of these rings every ideal is principal, generated by some non-negative integer.
**(4)** In $\mathbb{Z}$, any two distinct positive integers generate different ideals. Show that is not true in $R$, $S$, $T$. For each of these rings, identify a set of integers that uniquely define all ideals.
**(5)** Which of these ideals are prime?

*Problems* 9.2. Let $R$ be an integral domain and let $D$ be a multiplicatively closed set in $R$. Define a relation $\sim$ on $R \times D$ by

$$(a_1, d_1) \sim (a_2, d_2) \quad \text{when} \quad a_1 d_2 = a_2 d_1$$

**(1)** Show that this relation is an equivalence relation.
**(2)** Let $[a, b]$ denote the equivalence class of $(a, b)$. Call the set of equivalence classes $D^{-1}R$. Show that the function below is injective.

$$\begin{aligned} R &\longrightarrow D^{-1}R \\ r &\longmapsto [r, 1] \end{aligned}$$

*Problems* 9.3. Let $R$ be an integral domain and $D$ a multiplicatively closed subset. [If it helps, you get started, let $R = \mathbb{Z}$ and $D = \mathbb{Z} \setminus \{0\}$.] Using the relation in the previous problem, define addition and multiplication on $D^{-1}R$ by

- $[a_1, d_1] + [a_2, d_2] := [a_1 d_2 + a_2 d_1, d_1 d_2]$, and
- $[a_1, d_1] \star [a_2, d_2] := [a_1 a_2, d_1 d_2]$,

**(1)** Show that multiplication is well-defined. That is if $(a_1, d_1) \sim (b_1, f_1)$ and $(a_2, d_2) \sim (b_2, f_2)$ then
$$(a_1a_2, d_1d_2) \sim (b_1b_2, f_1f_2)$$

**(2)** (HW) Show that addition is well-defined. (This is the similar to the previous problem, but the computation is a bit more involved.)

**(3)** Identify the additive and multiplicative identities, and the additive inverse of an element.

**(4)** Prove that multiplication is commutative and associative.

**(5)** (HW) Prove that addition is commutative and associative. (Again, a bit more involved than for multiplication.)

**(6)** Prove distributivity holds.

*Problems* 9.4. Let $R$ be a ring and $D$ a multiplicatively closed subset.

**(1)** For $I$ and ideal in $R$ let $D^{-1}I = \left\{ \frac{a}{d} : a \in I, d \in D \right\}$. Show that $D^{-1}I$ is an ideal in $D^{-1}R$.

**(2)** (HW) For a prime ideal $P$ in $R$, $D^{-1}P$ is prime if and only if $D \cap P = \emptyset$. What is $D^{-1}P$ when $D \cap P \neq \emptyset$?

**(3)** Show that every ideal in $D^{-1}R$ is $D^{-1}I$ for some ideal $I$ in $R$. (Hints: If $J$ is an ideal in $D^{-1}R$ then an element of $J$ may be written $a/s$ for $a \in R$ and $s \in D$. Show that $D^{-1}(J \cap R) = J$.)

*Problems* 9.5. (HW) Fix a ring $R$ and a prime ideal $P \subseteq R$, and let $D = R \setminus P$.

**(1)** Prove that $D$ is a multiplicative set.

**(2)** Prove that the ring $R_P = D^{-1}R$ has a unique maximal ideal (the ring $R_P$ is called the *localization of* $R$ *at* $P$ and plays an important role in algebraic geometry).

*Problems* 9.6. (Challenge problem as an option to replace the previous problem.) Let $R$ be an integral domain. A multiplicatively closed set $S \subseteq R$ is *saturated* when

$$xy \in S \iff x \in S \text{ and } y \in S.$$

**(1)** There is a theorem saying $S$ is saturated iff $R \setminus S$ is a union of prime ideals. Prove one direction of this result: Let $\mathcal{P}$ be a set of prime ideals and let $S = R \setminus \Big( \cup_{P \in \mathcal{P}} P \Big)$. Show that $S$ is multiplicatively closed and saturated.

**(2)** Show that the set $S = \left\{ 300^i : i \in \mathbb{N} \right\}$ is multiplicatively closed. Find its saturation (the smallest saturated multiplicatively closed set containing $S$).