Problem set for Feb. 10: Problems from Axler Ch. 3

Key lemma (3.4): Given $v_1, \dots v_m$ a basis for V and any $w_1, \dots w_m$ in W, there is a unique $T \in \mathcal{L}(V, W)$ taking v_i to w_i.

Proposition (3.6): $\mathcal{L}(V, W)$ is a vector space.

For the following results, let $T \in \mathcal{L}(V, W)$.

Proposition (3.13): The nullspace of T is a subspace of V.
Proposition (3.18): The range of T is a subspace of W.

Theorem (3.21): $\dim\big(\text{null}(T)\big) + \dim\big(\text{range}(T)\big) = \dim(V)$.

Corollary (3.22): If $\dim(V) > \dim(W)$ then T is not injective ($\text{null}(T)$ is non-trivial).
Corollary (3.24): If $\dim(W) > \dim(V)$ then T is not surjective

(1) Proposition (3.6) says that $\mathcal{L}(V, W)$ is a vector space. Here we do some due diligence, and prove a couple of things. Be explicit about justifying each step.

- The first claim that must be shown is that the operations of addition and scalar multiplication are well-defined. We showed that the sum of two linear maps is also linear. Show that the product of a linear map by a constant is also linear.
- The second claim that must be shown is that the two operations satisfy all the vector space properties. There are many.
  Show that this distributive property holds: $a(S + T) = aS + aT$.

(2) We proved in class that the nullspace of a linear transformation is a vector space.
Prove that the range of $T \in \mathcal{L}(V, W)$ is a vector space.

(3) Find a basis for $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$.

(4) Let $T : U \longrightarrow V$ and $S : V \longrightarrow W$ be linear maps.

- Show that the composition $S \circ T$ is also a linear map. What do you need to show?
- It is a general fact about functions that $(f \circ g) \circ h = f \circ (g \circ h)$. Convince yourself of this.
- Prove the distributive law for linear maps: for $T_1, T_2 \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$:

$$S \circ (T_1 + T_2) = S \circ T_1 + S \circ T_2$$

- It is also the case that for ID the identity map on V (the book just uses I): $T \circ ID = T$.
- Henceforth we will generally not write $\circ$ in the composition of linear maps, and just write ST. Notice though, that the codomain of T must equal the domain of S for this to make sense.

(5) 3B #2 Let $S, T \in \mathcal{L}(V)$. Prove the following.

- $ST = TS$ is not always true. Find a simple counter example using $\mathbf{F}^2$.
- If the range of T is a subspace of the nullspace of S, then $ST = 0$
- If the range of S is contained in the nullspace of T then $(ST)^2 = 0$.