1 Chapter 3A-D in a nutshell

Assume all vector spaces U , V , W are finite dimensional over a field $\mathbf{F}$. References (3.x) are to Axler's book.

Linear transformations (also called maps)

Definition 1.1. A function $T : V \longrightarrow W$ is **linear** when it is additive ($T(v+v') = T(v)+T(v')$ for $v, v' \in V$) and homogeneous ($T(\lambda v) = \lambda V$ for $\lambda \in \mathbf{F}$ and $v \in V$). We also call these maps and transformations.

(Sec 3D) T is an **isomorphism** when it is linear and also bijective (injective and surjective). We showed in class that the inverse function of a linear map is also linear.

Lemma 1.2 ((3.4) Key lamma)**.** *Given $v_1, \dots v_m$ a basis for V and any $w_1, \dots w_m$ in W , there is a unique $T \in \mathcal{L}(V, W)$ taking v_i to w_i .*

Proposition 1.3 (3.6)**.** *The set of linear maps from V to W , denoted $\mathcal{L}(V, W)$, is a vector space.*

Theorem 1.4 (Fundamental Theorem of Linear Maps (3.13), (3.18), (3.21))**.** *Let $T \in \mathcal{L}(V, W)$.*

(1) The nullspace of T is a subspace of V.

(2) The range of T is a subspace of W.

(3) $\dim\big(\mathrm{null}(T)\big) + \dim\big(\mathrm{range}(T)\big) = \dim(V)$.

Corollary 1.5. *The following are equivalent for $T \in \mathcal{L}(V)$. (Or if $T \in \mathcal{L}(V, W)$ with $\dim(V) = \dim(W)$.)*

(1) T is injective.

(2) T is surjective.

(3) T is bijective (that is, T is invertible).

Coordinates and Matrices

(3.29-3.40) essentially give these two results.

Proposition 1.6. *A basis $\mathcal{V} = (v_1, v_2, \dots, v_n)$ for V determines a unique isomorphism from V to $\mathbf{F}^n$. It takes an arbitrary $v = a_1v_1 + \dots a_nv_n$ (which can be uniquely expressed in this way with $a_i \in \mathbf{F}$!) to the $n \times 1$ matrix* $\begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$.

Proposition 1.7. *Let $\mathcal{V} = (v_1, v_2, \dots, v_n)$ be a basis for V and let $\mathcal{W} = (w_1, w_2, \dots, w_p)$ be a basis for W .*

(1) There is a function $\mathcal{M}$ from $\mathcal{L}(V, W)$ to $\mathbf{F}^{p\times n}$ (the $p\times n$ matrices over $\mathbf{F}$ with component-wise addition).

(2) *This function takes* $T \in \mathcal{L}(V,W)$ *to the* $p \times n$ *matrix* $\mathcal{M}(T,\mathcal{V},\mathcal{W})$. *Calling this matrix* A *for the moment, the entries of* A *are* $A_{j,k}$ *where*

$$T(v_k) = \sum_{j=1}^{p} A_{j,k} w_j = A_{1,k} w_1 + A_{2,k} w_2 + \cdots + A_{p,k} w_p$$

(3) *This function* $\mathcal{M}$ *is uniquely defined by the equations above.*

(4) *The function* $\mathcal{M}$ *is an isomorphism: it is linear and bijective.*

Note: Since $\mathcal{W}$ is a basis, the $A_{j,k}$ are uniquely determined!

The composition of linear maps corresponds to the product of matrices (recall we are writing the composition of linear maps like a product, without the $\circ$ that is usually used for composition).

Proposition 1.8 (Composition of linear maps corresponds to multiplication of matrices)**.** *If* $T \in \mathcal{L}(U,V)$ *and* $S \in \mathcal{L}(V,W)$ *and we have bases* $\mathcal{U}$, $\mathcal{V}$ *and* $\mathcal{W}$ *for* U, V, *and* W, *respectively, then*

$$\mathcal{M}(ST,\mathcal{U},\mathcal{W}) = \mathcal{M}(S,\mathcal{V},\mathcal{W})\mathcal{M}(T,\mathcal{U},\mathcal{V})$$

Let's write the identity map on a vector space V as ID, and use I for the identity matrix (1s on the diagonal, 0s outside the diagonal). Then $I = \mathcal{M}(ID,\mathcal{V},\mathcal{V})$.

Exercises 1.9. These exercises require understanding how a nice choice of basis can give a simple matrix for a linear transformation.

(a) 3C # 2
(b) 3C #5, 6, 7
(c) 3D # 3
(d) 3D #11

Exercises 1.10. These exercises involve quick applications of the fundamental theorem of linear maps. Use complete sentences!

(a) 3C # 17
(b) 3D #2
(c) 3D # 3
(d) 3D #11

Exercises 1.11. These exercises involve understanding composition of linear maps and the fundamental theorem of linear maps, with particular attention to invertible maps

(a) 3D # 1
(b) 3D #2
(c) 3D # 3
(d) 3D #5, 6, 7 are a little harder.
(e) 3D #11, 12

Exercises 1.12. (3D #6) Let $S,T \in \mathcal{L}(V,W)$. Prove that $\text{null}(S) = \text{null}(T)$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

(a) Draw a diagram that illustrates the maps S,T and the proposed map E.
(b) Prove the backward direction: if the invertible map E exists, how do we know that $\text{null}(S) = \text{null}(T)$?

(c) For the forward direction, choose vectors $v_1, \ldots, v_r$ such that $T(v_i)$ are a basis for $\operatorname{range}(T)$. Do the same for S.

(d) Now define the map E. You will need Lemma 1.2 above, and the result from the last chapter that any linearly independent set can be extended to a basis.

The transpose of a $p \times n$ matrix A is the $n \times p$ A^t whose k, j entry is $A_{j,k}$. 3C# 14 asks you to show that this is a linear transformation. 3C#15 asks you to show $(AB)^t = B^t A^t$. It is worth doing because it forces you to use the formula for a matrix product carefully.

The transpose, rank of a matrix, change of basis

Here are two things, one from the end of 3C, the other from the end of 3D, that are worth careful attention in class.

Proposition 1.13. *Suppose A is an $p \times n$ matrix over $\mathbf{F}$.*

(1) The column rank of A equals the row rank of A.

(2) If that rank is r then there is a $p \times r$ matrix C and an $r \times n$ matrix R, both of rank r, such that $A = CR$.

Exercises 1.14. This example illustrates the essence of the proof of the proposition.

(a) Let A be the matrix below. Show that the column rank is two by finding two columns that span the column space.

$$\begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ 6 & 2 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) Use that observation to factor $A = CR$ where C has 2 columns.

(c) Use the factoriation to explain why the row rank of A is at most two.

(d) A generalization of this argument shows that the row rank of A is less than or equal to the column rank of A. Apply that result to A^t to get that the two ranks are equal.

Proposition 1.15 (Change of basis (3.82) (3.84))**.** *Suppose $\mathcal{U}$ and $\mathcal{V}$ are both bases for V.*

(1) Then $\mathcal{M}(ID, \mathcal{U}, \mathcal{V})\mathcal{M}(ID, \mathcal{V}, \mathcal{U}) = I$.

(2) Let $T \in \mathcal{L}(V)$. Then

$$\mathcal{M}(T, \mathcal{V}, \mathcal{V}) = \mathcal{M}(ID, \mathcal{U}, \mathcal{V})\mathcal{M}(T, \mathcal{U}, \mathcal{U})\mathcal{M}(ID, \mathcal{V}, \mathcal{U})$$

(3) Another way to see this is, letting $C = \mathcal{M}(ID, \mathcal{U}, \mathcal{V})$, if $B = \mathcal{M}(T, \mathcal{V}, \mathcal{V})$ and $A = \mathcal{M}(T, \mathcal{U}, \mathcal{U})$ then

$$A = C^{-1}BC.$$

The final two items of the proposition give "change of basis" formulas. Given the matrix for a linear transformation $T \in \mathcal{L}(V)$ in one basis, it shows how to get the matrix for T in a different basis.

Tuesday Problems

Prove one piece of Cor 1.5 above: If $T \in \mathcal{L}(V)$ is injective, then it is also surjective.

Prove 3D #11: Let $S, T \in \mathcal{L}(V)$. If ST is invertible then so are S and T.

Exercise 1.14 above.

Exercise 1.12 above. The solution is below.

Claim: Let $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null}(S) = \text{null}(T)$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

Suppose that the map invertible map E exists. Let $x \in \text{null}(T)$. Then $S(x) = ET(x) = E(T(x)) = E(0) = 0$. This shows that $\text{null}(T) \subseteq \text{null}(S)$. A similar argument using $T = E^{-1}S$ proves the reverse containment.

Suppose now that $\text{null}(S) = \text{null}(T)$. Proposition (2.33) in the book (or see my Ch 2 in a Nutshell Theorem) says that there is a subspace U in V such that $V = U \oplus \text{null}(S)$. Choose vectors $u_1, \ldots, u_r$ that form a basis for U Now let $w_i = S(v_i)$ and extend this to a basis $w_1, \ldots w_p$ for W (assuming $\dim(W) = p$). Similarly, let $w'_i = T(v_i)$ and extend this to a basis $w'_1, w'_2, \ldots, w'_p$ for W. Now Lemma 1.2 (3.4) tells us that we can define a linear map $E \in \mathcal{L}(W, W)$ that takes w_i to w'_i. I claim it is injective (and therefore bijective by Corollary 1.5).

To show E is injective, take an arbitrary $w \in W$, which, since $w_1, \ldots, w_p$ is a basis may be written in a unique way as $w = a_1 w_1 + \cdots + a_p w_p$. Apply E to get $E(w) = a_1 w'_1 + \cdots + a_p w'_p$. Suppose this is equal to 0. Since $w'_1, \ldots, w'_p$ are linearly independent, it must be the case that each $a_i = 0$. This tells us that $w = 0w_1 + \cdots + 0w_p = 0$. This shows E is injective by (3.15).