

Chapter 5A-D, 8A-C in a nutshell

1 Invariant spaces and eigenspaces

Let V be a vector space of dimension n over \mathbf{F} . We call a linear map from V to itself an **operator**. Chapters 5 and 8 are about operators, so maps from one vector space to a different one are not considered. In what follows, $T \in \mathcal{L}(V)$.

References (5.x, 8.x) are to Axler's book.

Invariant Spaces, Eigenvectors and Eigenvalues

Definition 1.1. A subspace U of V is **invariant** under T when $Tu \in U$ for all $u \in U$.

A vector v is an **eigenvector** of T when there is some $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$. In this case λ is the **eigenvalue** associated to v .

The **eigenspace** of an eigenvalue λ , written $E(\lambda, T)$, is the nullspace of $T - \lambda I$.

(Ch 8) The **generalized eigenspace** associated to λ , written $G(\lambda, T)$, is the nullspace of $(T - \lambda)^n$.

- The space spanned by an eigenvector v is an invariant subspace of T .
- The eigenspace associated to λ is also an invariant subspace of T .
- The generalized eigenspace associated to λ is also an invariant subspace of T .

Definition 1.2. Let $T \in \mathcal{L}(V)$.

- We say T is **diagonalizable** when there exists a basis \mathcal{V} for V such that $\mathcal{M}(T, \mathcal{V})$ is a diagonal matrix (all the non-diagonal entries are 0).
- We say T is **upper-triangularizable** when there exists a basis \mathcal{V} for V such that $\mathcal{M}(T, \mathcal{V})$ is an upper triangular matrix (all the entries below the diagonal are 0).

Chapter 5 gives conditions that T must satisfy to be upper-triangularizable (5C) and for it to be diagonalizable (5D). Chapter 8 takes a step further and shows that over \mathbb{C} , all operators can be put in a very special upper-triangular form that is called Jordan form. A key tool is the minimal polynomial of an operator.

2 The minimal polynomial

Theorem 2.1 ((5.22, 5.29)). *Let $T \in \mathcal{L}(V)$, with V of dimension n .*

- (1) *There exist polynomials $p(z) \in \mathcal{P}(\mathbf{F})$ such that $p(T) = 0$.*
- (2) *Among these polynomials that vanish at T , there is a unique one of minimal degree that is monic. Let's write it p_T .*
- (3) $\deg(p_T(z)) \leq n$.
- (4) *If $q(T) = 0$, then $p_T(z)$ divides $q(z)$.*

We call p_T the **minimal polynomial** of T . Note that the book doesn't use the notation, p_T .

Continuing with the notation of the theorem.

Corollary 2.2 ((5.31)). *If U is an invariant subspace of V then $T|_U$ is an operator on U (since it maps U to U). The minimal polynomial for $T|_U$ divides p_T .*

Corollary 2.3 ((5.27)). *If λ is an eigenvalue of T then $z - \lambda$ is a factor of p_T . Conversely, if $z - \lambda$ is a factor of p_T then λ is an eigenvalue of T .*

Corollary 2.4 ((5.27)). *Over the complex numbers, \mathbb{C} , the minimum polynomial factors completely: $p_T(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ where $\lambda_1, \dots, \lambda_m$ are eigenvalues of T (not necessarily distinct).*

Here the λ_i are not necessarily distinct.

Over the real numbers, \mathbb{R} , if $\dim(V)$ is odd then every operator on V has at least one eigenvalue.

Problems

Exercises 2.5. The minimal polynomial and invertible maps.

- (a) Show that T is invertible if and only if the constant term of p_T is nonzero.
- (b) Suppose that T is invertible. Show that if λ is an eigenvalue of T , then λ^{-1} is an eigenvalue of T^{-1} .
- (c) There is a simple relationship between p_T and $p_{T^{-1}}$. See if you can find it and prove it.
- (d) Suppose that T is invertible. Let $S \in \mathcal{L}(V)$. Show that the minimal polynomial for TST^{-1} is equal to the minimal polynomial for S .
- (e) Suppose that T is invertible. Let $S \in \mathcal{L}(V)$. Show that the minimal polynomial for TS is equal to the minimal polynomial for ST .

3 Operators that have an upper triangular matrix

Theorem 3.1 ((5.39), (5.44)). *Let $T \in \mathcal{L}(V)$. Let $\mathcal{V} = (v_1, \dots, v_n)$ be a basis for V . (TFAE) The following are equivalent.*

- (1) *The matrix of T in the basis \mathcal{V} is upper triangular.*
- (2) *$\text{span}(v_1, \dots, v_k)$ is invariant under T for each $k \leq n$.*
- (3) *$Tv_k \in \text{span}(v_1, \dots, v_k)$ for each $k \leq n$.*
- (4) *The minimal polynomial of T factors completely: $p_T(z) = (z - \lambda_1) \cdots (z - \lambda_k)$ for some $\lambda_i \in \mathbf{F}$ (not necessarily distinct).*

We will call such an operator **upper-triangularizable** (even though it is hard to say).

Corollary 3.2 ((5.40), (5.41)). *If T is upper triangularizable using \mathcal{V} , the diagonal entries of $\mathcal{M}(T, \mathcal{V})$ are the eigenvalues of T .*

Corollary 3.3 ((5.47)). *Over \mathbb{C} , the complex numbers, every operator (on a finite dimensional vector space) is upper-triangularizable.*

Exercises 3.4. Non upper-triangularizable operators.

- (a) Explain where to search (what field, what dimension) for the simplest example of a non upper-triangularizable operator.
- (b) Create one.
- (c) Thinking geometrically, characterize a whole bunch of linear maps that have no eigenvectors.
- (d) Give an example of an invertible matrix that has 0s on the diagonal (5C #4).
- (e) Prove or give a counterexample (5C #1): If $T \in \mathcal{L}(V)$ and T^2 is upper triangular with respect to \mathcal{V} then T has an upper matrix triangular with respect to some basis \mathcal{U} .

Exercises 3.5. Understanding the proof of Theorem 3.1: (1,2,3) implies (4). (The converse is more challenging and is proven by induction.) Recall that $p(T)q(T) = q(T)p(T)$ for polynomials p, q .

- (a) Show that a 2x2 upper triangular matrix A with diagonal entries α, β satisfies $(A - \alpha I)(A - \beta I) = 0$.
- (b) Do the analogous problem for a 3x3 upper triangular matrix A , but do it in a more sophisticated way: show that the standard basis vectors all get mapped to 0.
- (c) Suppose that \mathcal{V} is a basis in which T is upper triangular and that $\alpha_1, \dots, \alpha_n$ are the diagonal entries of $\mathcal{M}(T, \mathcal{V})$. Show that $(T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I) = 0$ (5.40), (5.41).

4 Operators that are diagonalizable

Theorem 4.1 ((5.54), (5.55), (5.62)). *Let $T \in \mathcal{L}(V)$.*

Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the distinct eigenvalues of T .

(TFAE) The following are equivalent.

- (1) *The matrix of T is diagonalizable.*
- (2) *V has a basis consisting of eigenvectors.*
- (3) *$V = E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \dots \oplus E(\lambda_m, T)$*
- (4) *$\dim V = \dim E(\lambda_1, T) + \dim E(\lambda_2, T) + \dots + \dim E(\lambda_m, T)$.*
- (5) *The minimal polynomial of T no repeated factors: $p_T(z) = (z - \lambda_1) \cdots (z - \lambda_m)$.*

We call such an operator **diagonalizable**.

Corollary 4.2 ((5.58)). *If T has $n = \dim V$ distinct eigenvalues then T is diagonalizable.*

Corollary 4.3 ((5.58)). *If T is diagonalizable and U is an invariant subspace of V then $T|_U$ is diagonalizable.*

Exercises 4.4. Pieces of the proof of Theorem 4.1.

- (a) Show that (1) and (2) are equivalent.
- (b) Show that (3) and (4) are equivalent: use (3.94).
- (c) Show that (2) implies (4). You might do this for a 5×5 matrix to see the reason.
- (d) Show that (4) implies (1) see also (5.11).
- (e) Show that (1) implies (5), by showing that the polynomial given sends every vector in a basis to 0. (The converse is by induction and is harder.)

Exercises 4.5. Nondiagonalizable matrices.

- (a) Give a 2×2 example of an upper triangular matrix that is not diagonalizable.
- (b) Give an example of a 3×3 matrix with a single eigenvalue that has a minimal polynomial of degree 3.
- (c) Can you generalize the previous example?

HW: 5C #2: use the formula (3.41) for the product of two matrices.

5C #3 Use what we proved about $p_{T^{-1}}$ in the problem 2.5(c)

5 Decomposing V into generalized eigenspaces

Let V be a vector space of dimension n over \mathbb{C} . In what follows, $T \in \mathcal{L}(V)$.

The first set of results shows that the nullspace of the powers of T increase, but eventually stabilize. This allows us to write T as a direct sum of two spaces related to T .

Proposition 5.1 ((8.1-4)). *Let V be a vector space of dimension n over \mathbb{C} , and $T \in \mathcal{L}(V)$.*

- (1) *The null space of T^k is contained in the nullspace of T^{k+1} .*
- (2) *If $\text{null } T^{m+1} = \text{null } T^m$ then $\text{null } T^{m+k} = \text{null } T^m$ for all $k > 0$.*
- (3) *For some $m \leq n$, $\text{null } T^{m+1} = \text{null } T^m$.*
- (4) *V is the direct sum of $\text{null } T^m$ and $\text{range } T^m$:*

$$V = \text{null } T^m \oplus \text{range } T^m$$

Recall that the generalized eigenspace associated to λ , written $G(\lambda, T)$, is the nullspace of $(T - \lambda I)^n$. (It may be the nullspace of a lower power, but by the n th power the sequence in the proposition has stabilized.)

Theorem 5.2 ((8.9), (8.20), (8.22)). *Let V be a vector space of dimension n over \mathbb{C} and $T \in \mathcal{L}(V)$. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the distinct eigenvalues of T .*

- (1) *$G(\lambda_k, T)$ is invariant under T for each $k = 1, \dots, m$.*
- (2) *$(T - \lambda_k I)$ restricted to $G(\lambda_k, T)$ is nilpotent for each $k = 1, \dots, m$.*
- (3) *$V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \dots \oplus G(\lambda_m, T)$.*
- (4) *T has a basis consisting of generalized eigenvectors. Using such a basis, \mathcal{V} , the matrix for T is block diagonal.*

$$\mathcal{M}(T, \mathcal{V}) = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$$

Exercises 5.3.

- (a) (8A #1) Suppose $\dim \text{null } T^4 = 8$ and $\dim \text{null } T^6 = 9$. Show $\dim \text{null } T^m = 9$ for all $m \geq 5$.

HW hint 8A #2. Suppose there is a linear combination that gives 0, and apply T^{m-1} . Etc.

6 Nilpotent operators

Now we look inside a particular generalized eigenspace to show that a well chosen basis produces a very simple form for the matrix representation.

Definition 6.1. An operator on V (of dimension n) is **nilpotent** when $T^n = 0$.

Referring to Proposition 5.1, if T is nilpotent that means $\text{null } T^n = V$ and $\text{range } T^n = 0$. Furthermore, the only eigenvalue of T is 0.

In Theorem 5.2, the operator $T - \lambda_i I$ when restricted to $G(\lambda_i, T)$ is nilpotent. So we now consider how choosing a basis for $G(\lambda_i, T)$ can make the block A_i in Theorem 5.2 have simple form. Let's simplify notation let S be a nilpotent operator (playing the role of $T - \lambda_i I$ on a vector space U (playing the role of $G(\lambda_i, T)$).

Proposition 6.2. Let $S \in \mathcal{L}(U)$. (TFAE) The following are equivalent.

- (1) S is nilpotent.
- (2) The minimal polynomial of S is z^m for some $m \in \mathbb{N}$.
- (3) There is a basis of S for which the matrix of s is upper triangular with 0 on the diagonal.

Theorem 6.3 ((8.45)). Let S be nilpotent on U . There is a basis for U in which the matrix of S is block diagonal, and every block has 0 entries everywhere except for 1 in positions $i, i + 1$. Here is a representative 4×4 block.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercises 6.4. Examples

- (a) Given 4 examples of 4×4 matrices that each have a unique eigenvalue such that the minimum polynomials have degree 1, degree 2, degree 3, and degree 4.
- (b) (8C#5) Suppose $T \in \mathcal{L}(\mathbb{C}^2)$ is defined by $T(w, z) = (-w - z, 9w + 5z)$. Find a Jordan basis for T .