Chapter 5A-D, 8A-C culminating thoughts and test prep.

1 Summary of the main theorems

Let V have dimension n and $T \in \mathcal{L}(V)$.

- **Minimum Polynomial:** There is a *unique* polynomial $p_T(z)$ that is monic, satisfies $p_T(T) = 0$ and divides all other polynomials $q(z)$ such that $q(T) = 0$.
- **5D Diagonalization:** If $p_T(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$ for *distinct* λ_i then there is a basis $\mathcal{V}$ for V such that $\mathcal{M}(T, \mathcal{V})$ is a diagonal matrix.
- If the minimum polynomial $p_T(z)$ factors completely (i.e. degree 1 factors, but not necessarily distinct) then
  - **5C Upper Triangularization:** There is a basis for V in which $\mathcal{M}(T, \mathcal{V})$ is upper triangular.
  - **8B Block Diagonalization:** Even better, there is a basis for V in which $\mathcal{M}(T, \mathcal{V})$ is block diagonal.
  - **8C Jordan Form:** Even better, there is a basis for V in which $\mathcal{M}(T, \mathcal{V})$ is block diagonal and each block is made of matrices in Jordan form.
- **Real problems:** Over $\mathbb{R}$, $p_T(z)$ may have quadratic factors. There are alternative versions of the above theorems in this case, but they are messier.

Diagonal means that the off diagonal entries are 0. The diagonal entries must be one of the λ_i, but some may occur more than once.

$$\mathcal{M}(T, \mathcal{V}) = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

Block diagonal means

$$\mathcal{M}(T, \mathcal{V}) = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$$

Jordan form means

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

2 In realistic problems, you start with a matrix

In practice, the linear operator T is usually given to us relative to some already chosen basis $\mathcal{E}$. That is, we are given a matrix

$$A = \mathcal{M}(T, \mathcal{E}).$$

But this basis is not the optimal one relative to getting the nice forms above (diagonal, block diagonal etc.). So the task in practice is to find a better basis. Note that the minimum polynomial for the matrix for T with respect to any basis is the same as p_T.

For example, let's suppose $\dim V = 5$ and $p_A(V) = (z-1)(z-2)(z-3)$ and the eigenspaces for 1 and for 3 have dimension 2, while the eigenspace for 2 has dimension 1. Then there are column vectors $x_1, x_2, y, w_1, w_2 \in F^5$ (these are in $\mathcal{E}$ coordinates) such that

$$Ax_i = x_i \qquad Ay_{=}2y \qquad Aw_i = 3w_i.$$

Check(!) that we can also write this as

$$A \begin{bmatrix} x_1 & x_2 & y & w_1 & w_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & y & w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Letting $X = \begin{bmatrix} x_1 & x_2 & y & w_1 & w_2 \end{bmatrix}$ be the matrix of the eigenvectors, and D the diagonal matrix of eigenvalues

$$\begin{aligned} AX &= XD \qquad\qquad \text{or, equivalently} \\ X^{-1}AX &= D \end{aligned}$$

The matrix X is called a **change of basis** matrix. In the basis formed by the columns of X the linear transformation T is simple. It is just given by a diagonal matrix.

For matrices that are not diagonalizable, a bit more is involved. The examples in the next section illustrate.

3 Practical problems you should be able to solve

Finding eigenvalues and vectors for a 2×2 matrix

Recall that we want to find a number λ and nonzero vector v such that $Tv = \lambda v$. Equivalently, we want to find a λ such that $T - \lambda I$ has a nonzero vector v in its nullspace. As above, let's assume that T is given with respect to some basis $\mathcal{E}$ and $A = \mathcal{M}(T, \mathcal{E})$. We need to find a λ such that $A - \lambda I$ is *not* invertible. This is a hard and messy problem, so we stick with 2×2 matrices and recall what you did in your first linear algebra class.

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its determinant, $ad - bc$ is nonzero. We want $A - \lambda I$ to not be invertible, so we set $\det(A - \lambda I) = 0$. Thus we solve $(a - \lambda)(d - \lambda) - cd = 0$. This is a quadratic in λ. We can do this! For larger n, the determinant has $n!$ terms, so is messy to compute. Even worse, solving a high degree polynomial is hard. (In fact it is impossible to get an exact solution to a general polynomial of degree ≥ 5.)

So here is a 2×2 matrix. Diagonalize A.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

This means

(1) Compute $\det(A - \lambda I)$;

(2) Find the roots λ_1, λ_2 of $\det(A - \lambda I)$;

(3) For each root λ_i, find a basis, v_i, for the nullspace of $A - \lambda_i$;

(4) Note that for $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ we have $AX = XD$. Equivalently, $X^{-1}AX = D$.

You can get more examples by asking an AI system: "Give me a 2 by 2 matrix with nonzero integer entries that has 2 distinct eigenvalues and the eigenvalues and eigenvectors have integer entries. Don't tell me how to diagonalize it yet. I'm going to do that and then ask if I did it correctly."

Finding the generalized eigenspace for a 2×2 matrix

The previous 2×2 matrix had distinct eigenvalues. Let's look at what can be done if there is only one eigenvalue, and just a one-dimensional eigenspace. Our canonical example is the matrix

$$K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalue is just 0, and there is one eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The generalized eigenspace is all of $\mathbb{R}^2$, and the vector $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector with this property: $Kx_2 = x_1$.

More generally for

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

There is one eigenvalue, λ, and there is one eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The generalized eigenspace is all of $\mathbb{R}^2$, and the vector $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector with this property: $Jx_2 = \lambda x_2 + x_1$. Written another way, $(J - \lambda I)x_2 = x_1$.

We want to mimic this for other 2×2 matrices that have a single eigenvalue, λ. That is, for a 2×2 matrix A with a single eigenvalue we want to find a matrix $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ such that

$$AX = \begin{bmatrix} \lambda x_1 & \lambda x_2 + x_1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Here are two examples that we will do in class. Find the eigenvalue λ for each. Find a basis for the nullspace, x_1. Find a vector x_2 such that $Ax_2 = \lambda x_2 + x_1$. Then "Jordanize" A. (Ditto for B and C.)

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

You can get more examples by asking an AI system: "Give me a 2 by 2 matrix, nonzero entries everywhere, with a unique integer eigenvalue. Don't tell me how to solve it yet. I'm going to put it in Jordan form and then ask if I did it correctly." You can also ask "What is the best method to find the basis to put a 2 x 2 matrix with one eigenvalue into Jordan form?"

Jordan form for larger matrices

How might we deal with larger matrices? Since finding the eigenvalues involves lots of computation, I have to do something to give you a manageable problem. One way is to give a block matrix. Another way is to give a matrix in upper triangular form, which is a bit more complicated. "Jordanize" these (a diagonal matrix is also considered to be in Jordan form).

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Exercises 3.1. Here are two other things to prepare for the test.

- **(a)** State a theorem similar to my Proposition 5.1 in the Ch5-8 nutshell notes that concerns the range, rather than the nulllspace.
- **(b)** Prove that theorem from first principles, (i.e. don't use the result of Proposition 5.1)
- **(c)** Give an example of a 2×2 matrix over $\mathbb{R}$ that is not Jordanizable over $\mathbb{R}$.