First Midterm Test: Thursday, March 5, 2026, 2:00-3:45
Do all problems. Please show your work! Good luck.
If you feel stuck on a problem write something to explain your thoughts on it.
There are 220 points possible, but the test is counted out of 200.

NAME: ____

1. [20 pts.] Show that the following set is not a subspace of $\mathbb{R}^3$ by *giving an example* where one of the properties of subspaces is violated.

$$\left\{x \in \mathbb{R}^4 : x_1 + x_2 + 3x_3 = 5\right\}$$

2. [30 pts.] State the main theorem on vector spaces of finite dimension from Chapter 2.

Here is the main theorem on linear maps from Chapter 3. You may use it freely in what follows.

Theorem (Fundamental Theorem of Linear Maps (3.13), (3.18), (3.21))**.** *Let* $T \in \mathcal{L}(V, W)$.

(1) The nullspace of T *is a subspace of* V*.*

(2) The range of T *is a subspace of* W*.*

(3) $\dim\big(\operatorname{null}(T)\big) + \dim\big(\operatorname{range}(T)\big) = \dim(V)$.

3. [30 pts.] Let $T \in \mathcal{L}(V)$. Explain why T is injective if and only if T is surjective.

4. [20 pts.] Factor the following matrix as $A = CR$ where C has two columns. Note that the column rank is two.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 2 \\ 4 & 1 & 2 & 3 & 5 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

5. [40 pts.] Let A be an $p \times n$ matrix with column rank r (the dimension of the column space).
Suppose $A = CR$ with C a matrix having r columns.
Show that the nullspace of R is equal to the nullspace of A.

6. [20 pts.] Let A be a $r \times p$ matrix and let B be a $p \times n$ matrix. Prove that $(AB)^t = B^t A^t$ by showing that the j, k component of these matrices are equal. (The superscript t indicates the transpose of a matrix.)

7 [30 pts.] Suppose that $S, T \in \mathcal{L}(V, W)$ and $E \in \mathcal{L}(V, V)$ is invertible.
Prove that $S = TE$ implies that $\text{range}(S) = \text{range}(T)$. [The converse is also true.]

8 [30 pts.] Let V and W be finite dimensional vector spaces. Let $T \in \mathcal{L}(V, W)$.
Suppose that $u_1, u_2 \in V$ are such that $T(u_1), T(u_2)$ form a basis for $\operatorname{range}(T)$.
Suppose also that v_1, v_2, v_3 are a basis for $\operatorname{null}(T)$.
Prove that u_1, u_2, v_1, v_2, v_3 are linearly independent. [They also span V.]