First Midterm Test: Tuesday, April 7, 2026, 2:00-3:45
Do all problems. Please show your work! Good luck.
If you feel stuck on a problem write something to explain your thoughts on it.
There are 220 points possible, but the test is counted out of 200.

NAME: ____

1. [45 pts.] For each property below give an example operator on $\mathbb{R}^2$ (i.e. a square matrix). Briefly explain and justify.

(1) A cannot be put in upper triangular form, but A^2 is diagonalizable (or, more challenging, A^3 is diagonalizable).

(2) B has diagonal entries that are 0 and has eigenvalues 1 and -1.

(3) C satisfies $\operatorname{null} C \cap \operatorname{range}(C) \neq \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

2. [50 pts.] Let V be a vector space of dimension n over $\mathbb{C}$, and $T \in \mathcal{L}(V)$.

(1) Prove that the range of T^k contains the range of T^{k+1}: $\operatorname{range} T^{k+1} \subseteq \operatorname{range} T^k$.

(2) Suppose $\operatorname{range} T^{m+1} = \operatorname{range} T^m$. Prove that $\operatorname{range} T^{m+2} = \operatorname{range} T^{m+1}$.
(This is the essence of an inductive proof that $\operatorname{range} T^{m+k} = \operatorname{range} T^m$ for $k > 0$.
You may write an inductive proof if you want extra credit.)

3. [75 pts.] Let $T \in \mathcal{L}(V)$ where V has dimension n. Suppose that T is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct), and associated eigenvectors $x_1, x_2, \ldots, x_n$.

(1) Explain what it means to say T is diagonalizable as stated above.

(2) What are the eigenvalues and eigenvectors of T^2? Justify your answer.

(3) What conditions on the λ_i ensure that T is invertible? Explain briefly.

(4) If T is invertible, what are the eigenvalues and eigenvectors of T^{-1}? Explain briefly.

(5) Suppose S is invertible. Can you say anything about the eigenvalues of STS^{-1}? Explain.

4. [50 pts.] Put the following matrix into Jordan form (it may be diagonal or not). That is, find a matrix X such that $AX = XJ$ where J is diagonal or of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.
Check that $AX = XJ$.

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$