From Ch 5: Let V have dimension n and $T \in \mathcal{L}(V)$.

Theorem ((5.44))**.** *The minimum polynomial of T factors completely as $(z-\lambda_1)\cdots(z-\lambda_m)$ $\iff$ T is upper triangularizable.*

The proof uses the fact that there is an eigenvector (choose any of the λ_i), then applies an inductive process to extend the basis. Later, in Ch 8 we showed something much stronger (Jordan form). Note: This is always true over $\mathbb{C}$, since all polynomials factor completely over $\mathbb{C}$.

Now let V be an inner product space

Theorem ((6.37))**.** *The minimum polynomial of T factors completely as $(z-\lambda_1)\cdots(z-\lambda_m)$ $\iff$ T is upper triangularizable using an orthonormal basis.*

You prove this by applying the Gram-Schmidt process.

Again, this is always true over $\mathbb{C}$.

Theorem ((7.27))**.** *Suppose that T is self-adjoint.* **All eigenvalues are real.** *The minimum polynomial factors completely.*

Theorem ((7.29))**.** *Suppose that T is self-adjoint. Let $\mathcal{E}$ be an orthonormal basis for which $A = \mathcal{M}(T, \mathcal{E})$ is upper triangular. Then A* **is actually diagonal**.

Theorem ((7.30))**.** *Suppose we are working over $\mathbb{C}$ and that T is normal. Let $\mathcal{E}$ be an orthonormal basis for which $A = \mathcal{M}(T, \mathcal{E})$ is upper triangular. Then A* **is actually diagonal**. *(Do this for $n = 3$.)*

Conversely, **if there is an orthonormal basis $\mathcal{E}$ for which $A = \mathcal{M}(T, \mathcal{E})$ is diagonal, then T is normal.**

Exercises 1.1.

(a) Prove the statements in bold above

Exercises 1.2. Consider $\mathbb{R}^n$ with the standard inner product. We will make a unitary (well, orthogonal) matrix Q and determine coordinates of a vector with respect to the orthonormal basis from the columns of Q.

Let $f_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $f_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

(a) Check that these are an orthonormal set and extend them to an orthonormal basis (i.e. find f_3).

(b) Use these as the columns to form a matrix Q. Show that $Q^* = Q^{-1}$. (So Q is unitary.)

(c) Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Compute $\langle v, f_i \rangle$.

(d) Show that $v = \sum_{i=1}^{3} \langle v, f_i \rangle f_i$.

Isometries and unitary matrices

Definition 1.3. Let V, W be inner product spaces of finite dimension. A transformation $S \in \mathcal{L}(V, W)$ is an **isometry** when $\|Sv\| = \|v\|$ for all $v \in V$.

Exercises 1.4. Let $S \in \mathcal{L}(V, W)$ be an isometry.

- **(a)** Show that S is injective, so dimension $V \leq \dim W$.
- **(b)** Show that if $V = W$ then the eigenvalues are all of norm 1.
- **(c)** Show that $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$. Hint look at $\|S(u + v)\|$. (You may assume $\mathbf{F} = \mathbb{R}$, an extra trick is needed for $\mathbb{C}$).
- **(d)** Use the previous to show that $S^*S = I$ (this is only possible if $\dim V \leq \dim W$).

We are particularly interested in isometries from V to itself, these are called **unitary** operators. We often use Q for a unitary matrix.

Exercises 1.5.

- **(a)** If Q is a unitary matrix then $Q^* = Q^{-1}$ since $Q^*Q = I$ and Q is square.
- **(b)** Show that for the identity operator I and two different bases for V, $\mathcal{E}$ and $\mathcal{F}$, the matrix $\mathcal{M}(I, \mathcal{E}, \mathcal{F})$ is a unitary matrix.
- **(c)** If $\mathcal{E}$ is an orthonormal basis for V and $\mathcal{F}$ is an orthonormal basis for W and $S \in \mathcal{L}(V, W)$ then the columns of $\mathcal{M}(S, \mathcal{V}, \mathcal{W})$ form an orthonormal list in $\mathbf{F}^m$.

Positive Matrices

Definition 1.6. An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and for all $v \in V$, $\langle Tv, v \rangle \geq 0$.

Exercises 1.7.

- **(a)** Suppose $T \in \mathcal{L}(V)$ and $T = R^*R$ for some $R \in \mathcal{L}(V)$. Show that T is a positive operator.
- **(b)** Suppose T is a positive operator. Since it is self-adjoint, the spectral theorem tells us that T is diagonalizable. More precisely, there is an orthonormal basis $\mathcal{E}$ for V consisting of eigenvectors for T.

$$\mathcal{M}(T, \mathcal{E}) = D$$

  1. Explain why D has nonnegative entries.
  2. Show there is a diagonal matrix R with nonnegative entries such that $D = R^2$.

- **(c)** Conclude: T is a positive operator if and only if $T = R^*R$ for some $R \in \mathcal{L}(V)$. Furthermore, we can assume R is diagonalizable with nonnegative eigenvalues.
- **(d)** Show the matrices below are positive operators and are invertible.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Singular Value Decomposition

We have shown that for any $T \in \mathcal{L}(V, W)$ that T^*T is a positive operator. In particular, since it is self-adjoint, by the spectral theorem, there is an orthonormal basis $\mathcal{E} = e_1, \dots, e_n$ for V consisting of eigenvectors for T^*T, with eigenvalues λ_i. Since T^*T is positive, the eigenvalues are nonnegative. Let's presume that the e_i are ordered so that $\lambda_i \geq \lambda_{i+1}$. Since the λ_i are nonnegative, they have real square roots, let's call them s_i: so $\lambda_i = s_i^2$. We call $s_1, \dots, s_n$ the **singular values** of T. There may be repeats, corresponding to the dimension of the associated eigenspace of T^*T and some eigenvalues can be 0.

Let R be the $n \times n$ diagonal matrix with the diagonal entries $s_1, \dots, s_n$. We have

$$\mathcal{M}(T^*T, \mathcal{E}) = R^2$$

Now read the proof of 7.70 carefully. Illustrating here with $n = 3$, and with $m = 2$ =the number of *positive* singular values. So e_3 is in the nullspace of T^*T and $s_3 = 0$. Let $f_1 = Te_1/s_1$ and $f_2 = Te_2/s_2$. These are vectors in W. The paragraph after 7.73 shows they are orthonormal (using the property of the adjoint!). Then Axler shows that for any $v \in V$ ($n = 3$, $m = 2$ here)

$$Tv = s_1\langle v, e_1\rangle f_1 + s_2\langle v, e_2\rangle f_2$$

Using matrices, this would seem to say that

$$\mathcal{M}(T, \mathcal{E}, \mathcal{F}) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

This is not quite correct, because V has dimension 3 and W may have a very large dimension (let's say dimension 4 as we continue our example). Since the range of T is just 2 dimensional, spanned by f_1 and f_2, the matrix above is correct when we consider T restricted to $(\operatorname{null} T)^\perp$ and mapping to $\operatorname{range} T$. To get the whole matrix for T, create an orthonormal basis for W by extending f_1, f_2 to $\mathcal{F} = f_1, f_2, f_3, f_4$. Then

$$\mathcal{M}(T, \mathcal{E}, \mathcal{F}) = \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Later in (7.77), we get the expression for T^*. This should be no surprise from the matrix above we have

$$T^*w = s_1\langle w, f_1\rangle e_1 + s_2\langle w, f_2\rangle e_2$$

BTW: we have made heavy use of the result that I had you exercise on Tuesday (last problem with $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$). Given an orthonormal basis $\mathcal{E}$ for V, the coordinates for $v \in V$ in the $\mathcal{E}$ basis are $\langle v, e_1\rangle, \dots, \langle v, e_n\rangle$.

Matrices and singular value decomposition

Here is how this relates to matrices. We are now using $V = \mathbf{F}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$) with the usual inner product, and the standard basis $e_1, \ldots, e_n$. Similarly $W = \mathbf{F}^k$ with the usual inner product and standard basis $f_1, \ldots, f_k$. Consider a $k \times n$ matrix A with rank $m \geq 1$. Note $\mathcal{E}$ and $\mathcal{F}$ are not the bases we want going forward (since A is probably not in diagonal form!). We will construct bases $\mathcal{C}$ for V and $\mathcal{B}$ for W in analogy to what we did above.

The $n \times n$ matrix A^*A is positive—equal to its conjugate transpose and satisfying $v^*A^*Av \geq 0$ for all $v \in \mathbf{F}^n$. (BTW it is clear that $v^*A^*Av \geq 0$ since $v^*A^*Av = \|Av\|$. (BTW, you can use this to shows that $\operatorname{null} A = \operatorname{null} A^*A$.)

There is some orthonormal basis $\mathcal{C} = c_1, \ldots, c_n$ consisting of eigenvectors for A^*A, ordered so that their eigenvalues satisfy $s_1^2 \geq s_2^2 \cdots \geq s_m^2 > 0$ and the eigenvalues for $c_{m+1}, \ldots, c_n$ are 0. Form the $n \times m$ matrix C, the $m \times m$ matrix R, and the $k \times m$ matrix B as follows.

$$C = \begin{bmatrix} c_1 & c_2 & \ldots & c_m \end{bmatrix} \qquad R = \begin{bmatrix} s_1 & 0 & \ldots & 0 \\ 0 & s_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & s_m \end{bmatrix} \qquad B = ACR^{-1}$$

Exercises 1.8. Prove the following claims

- **(a)** $B^*B = I_m$ the $m \times m$ identity matrix.
- **(b)** $AC = (BRC^*)C$
- **(c)** For c_j with $j > m$, $Ac_j = (BRC^*)c_j = 0$
- **(d)** Conclude that the following corollary holds.

Corollary 1.9. *There exists a factorization*

$$A = BRC^*$$

where B is a $k \times m$ matrix with orthonormal columns, C is an $n \times n$ matrix with orthonormal columns and R is an $n \times n$ diagonal matrix with strictly positive numbers on the diagonal.

Exercises 1.10. Exercises [Ax] 7E #1-5,7,9,10 and 7F (7.82), (7.85)

- **(a)** Prove s is a singular value iff there exist $v \in V$ and $w \in W$, such that $Tv = sw$ and $T^*w = sv$.
- **(b)** Find the singular values and the factorizations for $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ and $A' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$.
- **(c)** Show that $s_n \leq \|Tv\| \leq s_1$. See (7.85).
- **(d)** Show that T and T^* have the same positive singular values.
- **(e)** Suppose T is self-adjoint (so the spectral theorem applies and it is diagonalizable with some orthonormal basis). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. Show that the singular values are $|\lambda_i|$, after resorting to be in decreasing order. (7.21e).
- **(f)** Suppose T is invertible with singular values $s_1, \ldots, s_n$. Show that the singular values of T^{-1} are $1/s_n, \ldots, 1/s_1$ (in that order).