Gersgorins Theorem

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We begin by establishing some notation while giving a theorem from math 254 (without proof)

<u>Theorem 1</u>: For $A = [a_{ij}] \in M_n(F)$, the following are equivalent:

1. A is <u>nonsingular</u>. (i.e. $Ax = 0$ only when $x = 0$)
2. nullity$(A) = 0$
3. rank$(A) = n$
4. A is <u>invertible</u> (i.e., there exists $B \in M_n(F)$ with $AB = BA = I_n$)
4. det$(A) \neq 0$
5. RREF$(A) = I_n$
6. 0 is not an eigenvalue of A

<u>remarks</u>

i) Theorem 1 holds for any field F, but we are interested when $F = \mathbb{C}$ (or $\mathbb{R}$)

ii) In what follows, conditions 1 and 6 of theorem 1 will be most pertinent

$A \in M_n(\mathbb{C})$ is <u>diagonally dominant</u>
<u>in row i</u> if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

A is simply <u>diagonally dominant</u>
if A is diagonally dominant
in row i for all $i = 1, 2, \ldots, n$,

<u>remarks</u>

The above definition may also
be given as:

A is <u>diagonally dominant of rows</u>

or

A is <u>strongly diagonally dominant</u>

We say A is <u>weakly diagonally</u>
<u>dominant</u> (<u>of rows</u>) if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad , \text{all } i = 1, 2, \ldots, n$$

Theorem 2 (Levy-Desplanques) If $A \in M_n(\mathbb{C})$ is diagonally dominant, then A is nonsingular.

proof

We will show the contrapositive that if A is singular, then A is not diagonally dominant.

Suppose A is singular and $0 \neq x \in \mathbb{C}^n$ satisfies $Ax = 0$. We may divide x by one of its largest entries so we may assume that

$$1 = \max\{|x_1|, |x_2|, \cdots, |x_n|\} = x_i \quad .$$

Since $Ax = 0$, $(Ax)_i = 0$, or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0$$

Solving this equation for a_{ii} yields

$$a_{ii} = -\sum_{j \neq i} a_{ij}x_j ,$$

so that

$$|a_{ii}| \leq \sum_{j \neq i} |a_{ij}||x_j| \leq \sum_{j \neq i} |a_{ij}|$$

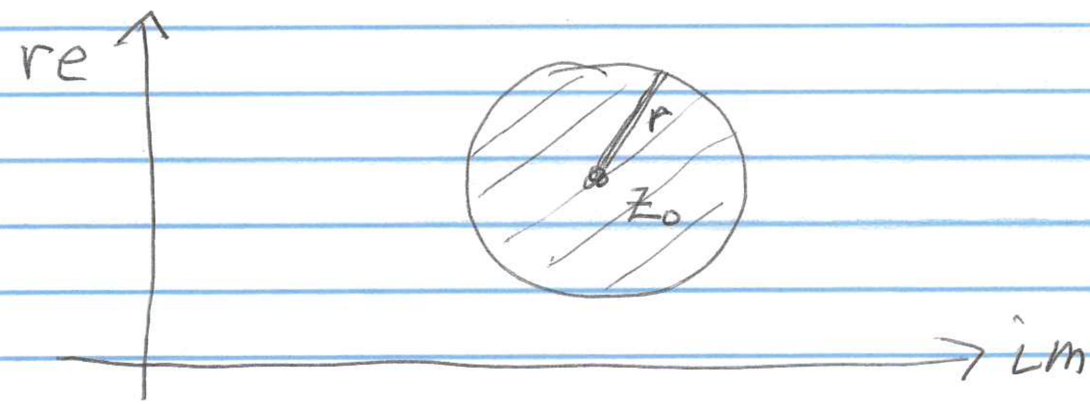
which shows that A is not diagonally dominant in row i. ∎

We will use theorem 2 to prove Gersgorins Theorem which comes next

For any $z_0 \in \mathbb{C}$ and $r \geq 0$, let

$$D(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| \leq r \}$$

be the <u>closed disc of radius r</u> with center z_0.



For any $A \in M_n(\mathbb{C})$, let

$$r_i = \sum_{j \neq i} |a_{ij}|,$$

and let

$$D_i = D(a_{ii}, r_i)$$

be the <u>Gersgorin Discs</u> of A, $i = 1, 2, \dots, n$.

Theorem 2 (Gersgorin) Every eigenvalue of A lies in $D_1 \cup D_2 \cup \cdots \cup D_n$.

proof

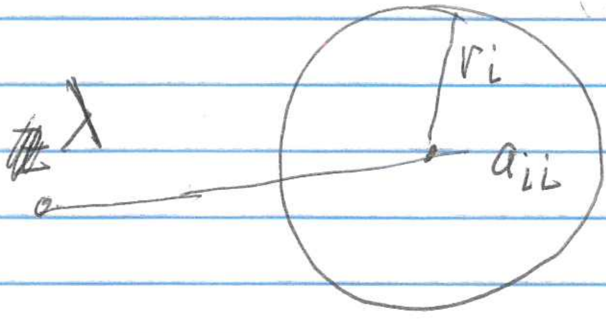
Suppose $\lambda \in \mathbb{C}$ and $\lambda \notin D_i$, $i = 1, 2, \cdots, n$. Recall that λ is an eigenvalue of A iff $\lambda I_n - A$ is singular. We claim λ is not an eigenvalue of A, or that $\lambda I_n - A$ is nonsingular

Since $\lambda \notin D_i$,

$$|\lambda - a_{ii}| > r_i = \sum_{j \neq i} |a_{ij}|$$

for all $i = 1, 2, \cdots, n$. This shows that $\lambda I_n - A$ is diagonally dominant and hence nonsingular.

Here we note that the Gersgorin Region $D_1 \cup D_2 \cup \cdots \cup D_n$ consists exactly of those λ for which $\lambda I_n - A$ is not diagonally dominant. ∎



remark: A and $\lambda I_n - A$ have the same r_i.

The last theorem needs to have some ideas from analysis, and "Gersgorins Theorem" frequently refers to theorems 3 and 4 together.

<u>lemma 1</u> Suppose $A(t)$, $0 \le t \le 1$ is a continuous function from $[0,1]$ to $M_n(\mathbb{C})$. Then there exists continuous <u>eigenfunctions</u>

$$\lambda_i : [0,1] \rightarrow \mathbb{C}$$

such that the eigenvalues of $A(t)$ are $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$.

Lemma 1 more or less follows from the following facts:

1. The coefficients of the characteristic polynomial are continuous functions of the entries of the matrix

and

2. The roots of a polynomial are continuous functions of its coefficients

The Gersgorin Region, $D_1 \cup D_2 \cup \cdots \cup D_n$ may be comprised of anywhere from 1 to n connected components.

For example, if A is 8-by-8 with Gersgorin Discs

then there would be four such components.
One of these is a single disc;
Two of these are unions of two discs;
and one of these is a union of three discs

<u>Theorem 4</u>: Let C be a connected component of $D_1 \cup D_2 \cup \cdots \cup D_n$ which is a union of k of the discs. Then C contains exactly k eigenvalues of A, counted according to algebraic multiplicity

proof sketch

Split $A = D + B$ where D is diagonal and B has zero diagonal

(example If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 0 \end{bmatrix}$.)

Define

$$A(t) = D + tB, \quad 0 \le t \le 1$$

so that $A(0) = D$, $A(1) = B$, and $A(t)$ is continuous (even linear!)

Let

$$\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$$

be the corresponding eigenfunctions

For each $0 \le t \le 1$, let $G(t)$ be the Gersgorin Region of $A(t)$, so that $G(t)$ contains the eigenvalues of $A(t)$.

For $t = 0$, $A(0) = D$, so the Gersgorin Discs all have radius 0 and the eigenvalues of $A(0)$ are just $a_{11}, a_{22}, \ldots, a_{nn}$.

In fact, we may assume that

$$\lambda_i(0) = a_{ii}, \quad \text{all } i = 1, 2, \ldots, n.$$

Assume that the connected component C_0 is a union of the first k Gersgorin Discs of A, $G = D_1 \cup D_2 \cup \cdots \cup D_k$.

For each $0 \leq t \leq 1$, the Gersgorin Discs of $A(t)$ have center a_{ii} and radius

$$t r_i, \quad \text{where } r_i = \sum_{j \neq i} |a_{ij}|$$

so that

$$G(0) \subseteq G(s) \subseteq G(t) \subseteq G(1) = G$$

whenever $0 \leq s \leq t \leq 1$.

Now, since $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ are continuous, none of these functions can "jump" from one component of G to another. Thus the first k eigenfunctions remain in C_0, and none of the other $n-k$ eigenfunctions can "jump in" to C_0.

exercise Use Gersgorin's Theorem to prove theorem 2 directly.

exercise Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

a) Find the Gersgorin region of A, G, and display G in the complex plane

b) Find and display the eigenvalues of A on the same page as for part a.

exercise Repeat the previous exercise for

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

exercise For $A \in M_n(\mathbb{C})$ let

$$\rho = \max_i \sum_{j=1}^{n} |a_{ij}|$$

be the maximum absolute row sum of A. If λ is an eigenvalue of A, show $|\lambda| \le \rho$.