Lecture Notes for Math 623
Matrix Analysis

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1 Hermitian Matrices

We can write any complex matrix as the sum of it's real part and imaginary part $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$, where $\mathrm{Re}(A), \mathrm{Im}(A)$ are both in $\mathcal{M}_n(\mathbb{R})$. We will see that there is a similar decomposition based on the eigenvalues of A.

Suppose that λ is an eigenvalue of A. Then λ is a root of the minimal polynomial of A (and conversely, if λ is a root of the minimal polynomial then λ is an eigenvalue of A). The minimal polynomial of A and of A^T are the same, so λ must also be an eigenvalue of A^T.

Taking conjugate transposes we can see that $\overline{\lambda}$ is an eigenvalue of A^*.

$$Au = \lambda u \iff u^* A^* = \overline{\lambda} u^*$$

Definition 1.1. A matrix $H \in \mathcal{M}_n$ is *Hermitian* when $H^* = H$, that is $\overline{h_{j,i}} = h_{i,j}$. A matrix is *skew-Hermitian* when $H^* = -H$. Over the reals, the conjugate is unnecessary and the terminology is different. A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is *symmetric* when $A^T = A$ and *skew symmetric* when $A^T = -A$.

Observations: There are a number of quick computations that reveal interesting properties of Hermitian matrices.

(1) A is Hermitian if and only if iA is skew Hermitian.

(2) The diagonal entries of a Hermitian matrix must be real. The diagonal entries of a skew-Hermitian matrix must be purely imaginary.

(3) The set of Hermitian matrices is closed under addition and scalar multiplication by a real number. So, the set of Hermitian matrices is a vector space over $\mathbb{R}$.

(4) The set of Hermitian matrices is not closed under multiplication. (Find A, B, both Hermitian, such that AB is not.)

(5) For any matrix A, $A + A^*$ and AA^* are Hermitian, and $A - A^*$ is skew-Hermitian.

(6) We may write A as the sum of a Hermitian matrix and a skew-Hermitian matrix

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

The final item was alluded to at the beginning of this section. It gives a decomposition of a matrix A into a Hermitian part and a skew-Hermitian part. The following proposition justifies thinking of this decomposition of A as a "real" part (the eigenvalues are real) and an "imaginary" part (the eigenvalues are pure imaginary).

Proposition 1.2. *The following properties hold for a Hermitian matrix A.*

(1) For all $v \in \mathbb{C}^n$, $v^*Av \in \mathbb{R}$.

(2) The eigenvalues of A are real.

(3) Let u, w be eigenvectors for distinct eigenvalues λ, μ, respectively. Then $v^*u = 0$.

(4) For any matrix S, S^*AS is Hermitian.

Proof. For any vector $v \in \mathbb{C}^n$,

$$(v^*Av)^* = v^*A^*v = v^*Av$$

so v^*Av must be real.

Let λ, u be an eivenvalue and associated eigenvector.

$$u^*Au = u * \lambda u = \lambda u^*u$$

By part (1) this must be real. Since $U * u$ is also real, λ must be real.

Let u, w be eigenvectors for distinct eigenvalues λ, μ, which by (2) must be real. Then $u^*Aw = u^*\mu w$, since but also $u^*Aw = \lambda u^*w$. Thus $\lambda u^*w = \mu u^*w$. Since $\lambda \neq \mu$, we must have $u^*w = 0$.

For any S, $(S^*AS)^* = S^*A^*S = S^*AS$, so S^*AS is Hermitian.

□

Exercises 1.3.

(a) Show that the eigenvalues of a skew-Hermitian matrix are pure imaginary.

(b) If A is Hermitian, then so is A^k for $k \geq 0$. If A Hermitian and also invertible A^k is Hermitian for all $k \in \mathbb{Z}$.

(c) Let A, B be Hermitian. Show that A and B are similar iff they are unitarily similar. [H1 4.1ex2].

(d) Let A, B be Hermitian.

(1) Show that $AB - BA$ is skew-Hermitian.

(2) Show that $\mathrm{Tr}((AB)^2) = \mathrm{Tr}(A^2B^2)$. [H4.1ex11] Hint: Consider $\mathrm{Tr}((AB - BA)^2)$.