Mike O'Sullivan
Department of Mathematics
San Diego State University Spring 2013

Math 623: Matrix Analysis
Schur's triangularization theorem

Theorem 0.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$. There is a real orthogonal matrix Q such that $Q^T AQ$ has the form*

$$\begin{bmatrix} B_1 & * & * & \dots & * & * \\ 0 & B_2 & * & \dots & * & * \\ 0 & 0 & B_3 & \dots & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & B_s \end{bmatrix}$$

where each B_i is either an eigenvalue of A (a 1×1 matrix), or $B_i = R^{-1}C(\lambda)R$ for a non-real eigenvalue λ of A.

Proof. We proceed by induction, assuming the statement of the theorem is true for square matrices of dimension less than n. Let $A \in \mathcal{M}_n(\mathbb{C})$. If A has a real eigenvalue λ, with eigenvector u, we can proceed as in the complex case by extending to an orthonormal basis $u, v_2, \dots, v_n$ and applying the induction hypothesis.

Suppose that A has no real eigenvalues, and let $\lambda, \bar{\lambda}$ be a conjugate pair of eigenvalues. From the observation we made in class,

$$A \begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} = \begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} \begin{bmatrix} \operatorname{Re} \lambda & \operatorname{Im} \lambda \\ -\operatorname{Im} \lambda & \operatorname{Re} \lambda \end{bmatrix} = \begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} C(\lambda)$$

Note that $\operatorname{Re} u$ and $\operatorname{Im} u$, may not be unit length (which is easy to fix), but more importantly, they may not be orthogonal. We can apply Gram-Schmidt. Let R be an upper triangular matrix such that

$$\begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} R = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

is an orthogonal pair of unit length vectors. Then

$$A \begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} R = \begin{bmatrix} \operatorname{Re} u & \operatorname{Im} u \end{bmatrix} RR^{-1}C(\lambda)R$$
$$A \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} R^{-1}C(\lambda)R$$

Now extend u_1, u_2 to an orthonormal basis, $u_1, u_2, v_3, \dots, v_n$. We then have

$$A \begin{bmatrix} u_1 & u_2 & v_3 & \dots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & v_3 & \dots & v_n \end{bmatrix} \begin{bmatrix} R^{-1}C(\lambda)R & X \\ 0 & B \end{bmatrix}$$

where B is $(n-2) \times (n-2)$, X is $2 \times (n-2)$ and the 0 stands for an $(n-2) \times 2$ matrix of 0s.

Applying the induction hypothesis to B gives the result.

□