Number Theory

Math 522, Fall 2000
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Computational Exercises: Rosen, *Elementary Number Theory and Its Applications* 4th ed.

1. §1.2 PP # 1 The tower of Hanoi puzzle.

2. §1.3 Fibonacci numbers and their ilk. Given g_1, g_2, a and b:
   a) Generate the sequence defined by $g_n = ag_{n-1} + bg_{n-2}$.
   b) Find the explicit definition of g_n as a function of n.
   c) Check that the explicit definition agrees with the recursive definition.

3. §2.1,2 Base b representations:
   a) Convert from base b to base 10 and vice-versa.
   b) Convert from base b to base b^r.

4. §3.1 Prime numbers:
   a) PP #4 Verify Goldbach's conjecture.
   b) CE #5 3 Compute twin primes.
   c) Use `nextprime[]` to compute $p/\ln p$ for the first 1000 primes.
   d) Graph $(n, p_n/\ln p_n)$ where p_n is the nth prime. Explain your result.

4. §3.3 The Euclidean Algorithm: Given a, b
   a)Find the greatest common divisor of a and b.
   a) Write the greatest common divisor as a linear combination of a and b using the Euclidean algorithm and report the number of steps it takes.
   b) Compare your results with Lamés Theorem.
   c) Compare your linear combination with what should be obtained according to the Theorem given in class.
   d) Write the greatest common divisor as a linear combination of a and b using the least remainder algorithm and report the number of steps it takes. Compare with the Euclidean algorithm.
   e) Extend these algorithms to find the g.c.d. of $a_1, \ldots, a_r$.

5. §3.4 Unique factorization:
   a) CE #2 Compare the number of primes less than n which are $1 \mod 4$ with the number which are $3 \mod 4$.
   b) Extend this to primes of the form $b \mod m$.
   c) CE #3 Find the smallest prime congruent to $b \mod m$.
   d) PP #2,3 Find the g.c.d. and l.c.m. of a, b from their prime factorizations. Extend to $a_1, \ldots, a_r$.
   d) PP #1 Find all of the divisors of n from its prime factorization.

6. §3.6 Linear Diophantine Equations:
   a) PP #1 Find the solutions of a linear diophantine equation in 2 variables.
   b) PP #2 Find the positive solutions.
   c) CE #1 For given a, b find all linear combinations $ax + by$ with x and y nonnegative.

7. §4.1,2 Modular arithmetic:
   a) PP §1#4 Experiment with efficient ways to perform modular exponentiation.
   b) PP §2#3 Compute inverses $\mod n$.
   c) PP §2#1,2 Solve linear congruences $\mod n$.

8. §4.3 The Chinese remainder theorem:
   a) Solve systems of congruences with coprime moduli using the Chinese remainder theorem.
   b) Now try it when the moduli are not coprime.

9. §4.5 Systems of linear congruences.
   a) Invert a 2×2 matrix over $\mathbb{Z}/n$.
   b) Solve a system of congruences over $\mathbb{Z}/n$.
   c) Extend to systems in n equations and n unknowns.

10. Ch. 5 Applications:
    a) Schedule round-robin tournaments for n teams.
    b) Assign a home team for each game in the case where n is odd.
    c) Write a hashing function for Social Security numbers for m students and $n > m$ memory locations.
    d) Experiment with your hashing function. How large should n/m be to make it rare for there to an instance where more than three probes are necessary for a success.
    e) Compute the parity bit for a bit string. Check whether an encoded string has an even or odd number of errors.
    f) Compute the check digit for and ISBN number. Check whether an ISBN number has an error.

11. Ch. 8 Cryptography:
    a) Encrypt and decrypt using an affine transformation modulo n.
    b) Encrypt and decrypt using an affine matrix transformation modulo n (a Hill cipher).
    c) Encrypt and decrypt using an exponentiation cipher.
    d) Encrypt and decrypt using the RSA cryptosystem.