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Math 627A: Modern Algebra I

Final Exam Preparation

On the final exam you will be asked to write an essay on TWO of the FOUR topics listed below. You have a certain degree of freedom in your essay, but I will ask a few specific questions on material that is central to the topic.

Please consider the following general guidelines:

- Break your essay into two parts.
  - Part 1: Imagine you are talking to a person who has a strong undergraduate background in Abstract Algebra (*e.g.* 521a/b). Your goal is to explain to them the essential topics in the course related to the given topic.
  - Part 2: Imagine you are talking to a fellow student in the course, or to a faculty member. Explain an advanced topic. This might be a theorem that we did not cover in the course, a challenging example, something you discovered in reviewing during the week, something you read in another source, etc.
- The challenge problems on the next page MAY be used for Part 2. They are NOT required, but I find them interesting.
- State theorems and definitions carefully and precisely when they are key elements of your essay. You may make reference to theorems using a shorthand, descriptive name, and you can create your own descriptor for the purposes of the essay (*e.g.* Unique Factorization Theorem, Root-Factor Theorem).
- Enrich your presentation with examples!
- Do not use your notes or a book.

Essay Topics

Topic A: The Fundamental Theorem of Galois Theory.

Topic B: Geometric constructions and the relationship to field theory.

Topic C: Solution of equations, cyclotomic polynomials and radical extensions.

Topic D: Fields in finite characteristic.

Challenge Problems:

Problem 1: Suppose that the polynomial $f = x^4 + ax^2 + b$ in $\mathbb{Q}[x]$ is irreducible. Let E be the splitting field of f and let the roots of f be $\pm\alpha$ and $\pm\beta$. We treated the case where $\alpha\beta \in \mathbb{Q}$ in class. Consider the extension of $\mathbb{Q}$ by a root γ of $y^2 + ax + b$.

- **(a)** Show that when $\alpha\beta \in \mathbb{Q}[\gamma]$, the Galios group of E is $\mathbb{Z}/4$. The polynomial $x^2 + 4x + 2$ is an example.
- **(b)** If $\alpha, \beta \notin \mathbb{Q}[\gamma]$ then the Galois group is D_4 the symmetry group for the square. The polynomial $x^4 - 2$ is an example. Identify the lattice of subgroups of D_4 and the corresponding lattice of intermediate fields between E and $\mathbb{Q}$.

Problem 2: In the Euclidean plane there is given a line segment of length 1 and the parabola T described by the equation $y = x^2$. Assume that we allow for the usual compass and straight-edge constructions, and in addition we allow for intersections of circles and lines with T.

- **(a)** Investigate intersections of T with lines and circles.
- **(b)** Describe a construction of the number $\sqrt[3]{2}$.
- **(c)** Determine those integers $m \in \mathbb{Z}$ for which $\sqrt[m]{2}$ can be constructed.
- **(d)** Can you determine other constructions that were not possible with ruler and compass?

Problem 3: Let $\phi_n(x)$ be the irreducible polynomial for a primitive n th root of unity. We found cyclotomic polynomials over the rationals $\phi_p(x)$ for p prime.

- **(a)** What is the degree of the cyclotomic polynomial $\phi_n(x) \in \mathbb{Q}[x]$ for arbitrary n? Can you find a formula for it?
- **(b)** Find $\phi_n(x)$ for $n \leq 16$.
- **(c)** What happens over a prime field? Are the polynomials $\phi_n(x)$ still irreducible?
- **(d)** How do roots of unity and subfields of finite fields interact?