Lecture Notes for Math 627B
Modern Algebra
The union of disjoint varieties

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1 Definition and key properties of modules

Theorem 1.1. *Let* $\mathbb{V}(I) = V_1 \cup V_2$ *and suppose that* $V_1 \cap V_2 = \emptyset$. *There are ideals* J_1 *and* J_2 *such that* $V_i = \mathbb{V}(J_i)$ *and* $I = J_1 J_2 = J_1 \cap J_2$.

Proof. Let $J_1 = I : \mathbb{I}(V_2)$ and $J_2 = I : \mathbb{I}(V_1)$. By the theorem on quotient ideals, $\mathbb{V}(J_i) = V_i$ since, for example, $\mathbb{V}(J_1) = \mathbb{V}(I) \setminus \mathbb{V}(\mathbb{I}(V_2))$.

Since $V_1 \cap V_2 = \emptyset$, J_1 and J_2 are comaximal. Thus $J_1 J_2 = J_1 \cap J_2$.

It is clear that $I \subseteq J_1 \cap J_2$, for in general, $I \subseteq I : K$. To prove the reverse containment, let $f \in J_1 \cap J_2$. Since J_1 and J_2 are comaximal, there exist $a \in J_1$ and $b \in J_2$ such that $a + b = 1$. Since $J_i \subseteq \mathbb{I}(V_i)$, we have $fa \in (J_1 \cap J_2)\mathbb{I}(V_1) \subseteq J_2\mathbb{I}(V_1) \subseteq I$, by the definition of J_2. Similarly, $fb \in I$, so $f = fa + fb \in I$. This shows $I = J_1 \cap J_2$ as claimed. □