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Math 627A: Modern Algebra I
Review for First Exam

Finite Fields

The problems on finite fields in my lecture notes are good preparation. Be able to compute with finite fields. Know the structure of finite fields (existence of a primitive element, subfields, automorphisms).

Groups Please read the following problems and their solutions in Ash's text. Many of them are routine, and others have been covered in class to some extent. Ash also provides solutions. (**Note**: Ash writes D_n as D_{2n}.)

- §1.1 pr. 2-11.
- §1.2 pr. 1-8.
- §1.3 pr. 1-12.
- §1.4 pr. 2-9.
- §1.5 pr. 1-8.

Let G be a group. The following groups and subgroups are important. You should be able to establish these results.

- $Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$ is a normal subgroup of G.
- The centralizer of $a \in G$, $C(a) = \{g \in g : ga = ag\}$ is a subgroup of G containing a. $Z(G) = \bigcap_{a \in A} C(a)$.
- Let H be a subgroup of G. The normalizer of H, $N_H = \{x \in G : x^{-1}Hx = H\}$ is a subgroup of H containing H. H is normal in N_H. Any subgroup K of G that contains H as a normal subgroup is contained in N_H. (If $N \trianglelefteq K \leq G$ then $K \leq N_H$.)
- The set of automorphisms of G, $\mathrm{Aut}(G)$, is a group.
- The set of inner automorphisms of G, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
- If G is *abelian*, $\mathrm{Tor}(G) = \{a \in G : \mathrm{ord}(a) \text{ is finite }\}$ is a normal subgroup of G and $G/\mathrm{Tor}(G)$ has no elements of finite order.

Problem: Shorties.

(a) Show that the intersection of two normal subgroups of G is normal in G.

(b) If every element in G has order 2 show that G is abelian.

(c) If G has even order then G has an element of order 2.
(Consider the pairing of g with g^{-1}).

Problem: Suppose G is abelian and $f : G \to \mathbb{Z}$ is surjective. Let K be the kernel. Show G has a subgroup H isomorphic to $\mathbb{Z}$ and $G \cong H \oplus K$.

Problem: Consider the following subgroups of $\mathrm{Gl}(F, n)$:

$$\mathrm{Sl}(F, n) \qquad \mathrm{Diag}(F, n) \qquad F^* I_n$$

(a) Show that $\mathrm{Sl}(F, n)$ and $F^* I_n$ are normal in $\mathrm{Gl}(F, n)$ but that $\mathrm{Diag}(F, n)$ is not, except in one very special case.
(b) Is it true that $\mathrm{Gl}(F, n)$ is the product of $\mathrm{Sl}(F, n)$ and $F^* I_n$? The answer is subtle—it depends on the field and on n!
To get started you might want to try some small fields, like $\mathbb{F}_3$, using a computer algebra system.