Math 241: Workshop on Geometry Software

Days 7 and 8, 2015-09-29, 2015-10-01

Trigonometric Functions

As a warmup, consider this construction.
Construction of points on the hyperbola $f(x) = 1/x$.
Given a point on the x-axis, $(a, 0)$, construct the following:

1. The vertical line, m, defined by $x = a$.
2. The line, h, defined by $y = 1$.
3. The intersection point $P = (a, 1)$ of m and h.
4. The line, k, from the origin to P.
5. The line, n, defined by $x = 1$.
6. The intersection point, Q, of k and n.
7. The horizontal line, o, through Q.
8. The point, Z, intersection of o and m.

The line k has equation $y = x/a$ because it goes through the origin and $(a, 1)$.
The coordinates of Q are $(1, 1/a)$ because it is on k and has x-coordinate 1.
The coordinates of Z are $(a, 1/a)$, since it is the intersection of $x = a$ and $y = x/a$.

We could also justify this construction using similar triangles.

Next we did our first trigonometric construction.
Construction of points $(t, \sin(t))$.

1. Circle c center $\mathcal{O} = (0, 0)$, radius 1.
2. Point $P = (1, 0)$ intersection of c with the x-axis.
3. Point Q on c.
4. Circular arc t with center $\mathcal{O}$ from P to Q.
5. Line $m : x = t$.
6. Line h perpendicular to the x-axis through Q.
7. Point Z intersection of m and h.

Assignment
(1) Create a worksheet that graphs both the sine and cosine functions. The directions for sine are given above; the directions for cosine require one additional step. You have to turn the a horizontal distance (the x-coordinate of Q) into a a vertical distance. Use a relection about some line or a rotation. List the construction steps and give some justification for the construction (like I did for $f(x) = 1/x$.)

(2) Create a worksheet for the graph of the tangent function. The first 4 steps are identical to the steps above for the sine function. Use the fact that $\frac{\tan(t)}{1} = \frac{\sin(t)}{\cos(t)}$.

Sums and products of trigonometric functions

As preparation we graphed $f(x) = M\sin(wx - a)$ where M, W, a are parameters (sliders in GeoGebra). The parameter M is called the *amplitude* and w is the *angular frequency*. The *phase shift* is a/w; think of rewriting the function as $M\sin(w(x - a/w))$.

We explored varying each parameter to see how each affects the graph. Next we added $f(x)$ to the standard $\sin(x)$ to explore what the sum of two sinusoidal functions looks like, and how the parameters affect the sum. We also experimented a bit with the sum of three or four sinusoids, and the period of the result. Then we looked at the product of two sinusoids, and again fiddled with parameters.

We reviewed this trig identity:

$$\sin(x)\sin(wx + a) = \frac{1}{2}\cos((w-1)x + a) - \frac{1}{2}\cos((w+1)x + a)$$

We referred to portions of the Wikipedia webpages on Amplitude Modulation and Heterodyne, in particular the paragraph "Simplified analysis of standard AM."

Assignment Show a simple AM implementation.

Let $c(x) = \sin(x)$ be a *carrier wave*, and let $m(x) = M\cos(wx+a)$ be the *message signal*. As the wikipedia page says, we should make $M < 1$ and the frequency w much smaller than the frequency of the carrier, which is 1. Let's take $0 < w < 0.2$ (in reality it is much much smaller than the carrier frequency).

The transmitted signal that "encodes the message onto the carrier" is $T(x) = (1 + m(x))c(x)$. Graph m, c, T. Show that the transmitted signal $T(x)$ is also equal to a sum of 3 sine waves.

$$T(x) = c(x) + \frac{M}{2}\sin(x(1+w) + a) + \frac{M}{2}\sin(x(1-w) - a)$$

Use color and other highlighting. If you feel adventurous play with the message $m(x)$: change the parameters to see how the transmitted signal changes; make $m(x)$ the sum of two cosines; or animate the parameters in $m(x)$.