Problem Set 12

Problems with (HW) are due Friday 12/9. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 12.1. (Discussion) Minimal polynomials and field extensions.

(a) Find the minimal polynomials of $3\sqrt{2}+4$, and $\sqrt[3]{2}+1$ over $\mathbb{Q}$. (Look for a quick and clever solution.)

(b) Find the minimal polynomial of $\alpha = \sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$. (Find a linear combination of $\alpha^4, \alpha^2, \alpha^2, \alpha, 1$ that adds to 0.)

(c) Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. (Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$ and get a contradiction.)

(d) Argue that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$.

(e) (HW) Find all 3 proper subfields of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ (aside from $\mathbb{Q}$).

(f) (HW) Parts (c) to (e) don't hold if 2 and 3 are replaced with arbitrary distinct positive integers m, n? What additional property might be needed?

Exercises 12.2. (HW) Splitting fields.

(a) Find the splitting field of $(x^2-5)(x^2-7)$ and all its subfields (3 proper subfields).

(b) Find the splitting field of x^4-4 over $\mathbb{Q}$ and all its proper subfields.

(c) Find the splitting field of x^d-1 over $\mathbb{Q}$ for $d = 2, 3, 4, 5,$ and 6.

(d) Conjecture a formula in $d \geq 2$ for the degree of the splitting field of x^d-1 over $\mathbb{Q}$.

Exercises 12.3. (HW) We studied the splitting field of x^4+1 over $\mathbb{Q}$ showing that it is $\mathbb{Q}(\alpha)$ where $\alpha = \frac{\sqrt{2}}{2}(1+i)$.

(a) Factor x^4+1 in $\mathbb{Q}(\alpha)$.

(b) There are 3 different ways to write x^4+1 as a product of quadratics (using complex coefficients). Find them.

(c) There are 3 different subfields of $\mathbb{Q}(\alpha)$. Find them.

(d) Explain the relationship between the factorizations of x^4+1 that you found in (b) and the fields you found in (c).

Exercises 12.4. We found the splitting field of x^3-2 over $\mathbb{Q}$ in class.

(a) This field has four proper subfields besides $\mathbb{Q}$ itself. Find them and find their dimensions over $\mathbb{Q}$.

(b) (HW) Factor x^3-2 over each of these subfields.